Faculty of Engineering
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Approximation Algorithms
Vertex Cover Problem

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Presented to:

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Outline

1. Review on Approximation Algorithms.
2. Vertex Cover Problem.
3. Approximation of VC problem.
4. AVC is a 2-polynomial approximation algorithm.

Review on Approximation Algorithms

- Some algorithms are NP-complete, so we don’t know how to find an optimal solution in polynomial time.

- **Solutions:**
  - If **input size is small**, the exponential algorithm is perfectly OK.
  - Isolate **important special cases** that we can solve in polynomial time.
  - Find **near-optimal solutions** in polynomial time through approximation algorithms.

- **Fact:**
  - Near optimality is good enough in most of the cases.

Review on Approximation Algorithms

- **C** is the cost produced by the approximate algorithm.
- **C*** is the cost produced by the optimal algorithm.
- **Assume that** the cost is always positive.

Two types of optimization problems:

Maximization problem:

$$0 \le C \le C^*$$

Minimization problem:

$$0 \le C^* \le C$$

Review on Approximation Algorithms

- We say that the algorithm of a given problem has an approximation ratio $\rho(n)$ if the cost of the solution **C** produced by the algorithm is of **factor** $\boldsymbol{\rho(n)}$ of the cost **C*** of the optimal solution.
- Approximation ratio $\rho(n) = \max\left(\frac{C^*}{C}, \frac{C}{C^*}\right)$
- Approximation ratio $\rho(n) \geq 1$

Maximization problem:

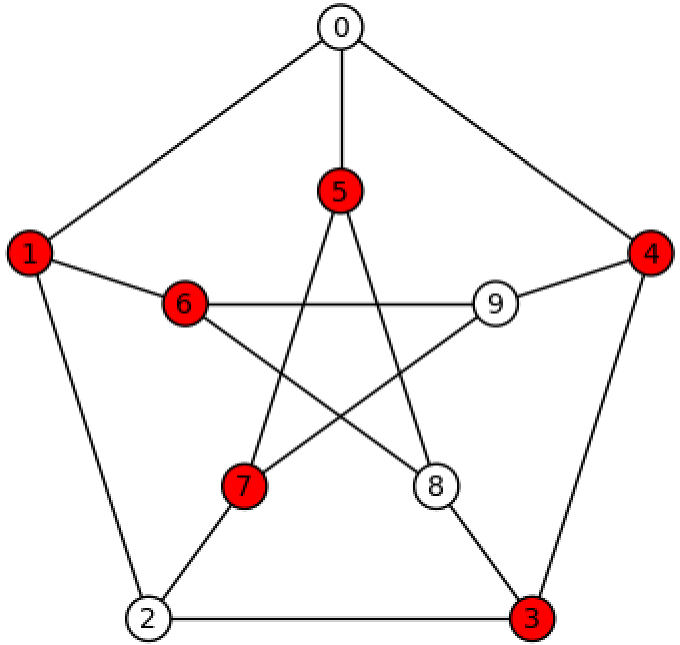
$$\mathbf{0} \leq \boldsymbol{C} \leq \boldsymbol{C}^*$$

Minimization problem:

$$\mathbf{0} \leq \boldsymbol{C}^* \leq \boldsymbol{C}$$

Vertex Cover Problem

- A **vertex cover** of an undirected graph **G = (V, E)** is a subset **V'** $\subseteq$ **V** such that if ***(u,v)*** is an edge of G, then either ***u*** $\in$ ***V'*** or ***v*** $\in$ ***V'*** (or both).
- The size of a vertex cover is the number of vertices in it.
- The **vertex-cover problem** is to find a vertex cover of minimum size in a given undirected graph.
- The vertex cover problem is **NP-complete**.



Vertex Cover Problem

- A **vertex cover** of an undirected graph **G = (V, E)** is a subset **V' ⊆ V** such that if ***(u,v)*** is an edge of G, then either ***u*** ∈ ***V'*** or ***v*** ∈ ***V'*** (or both).
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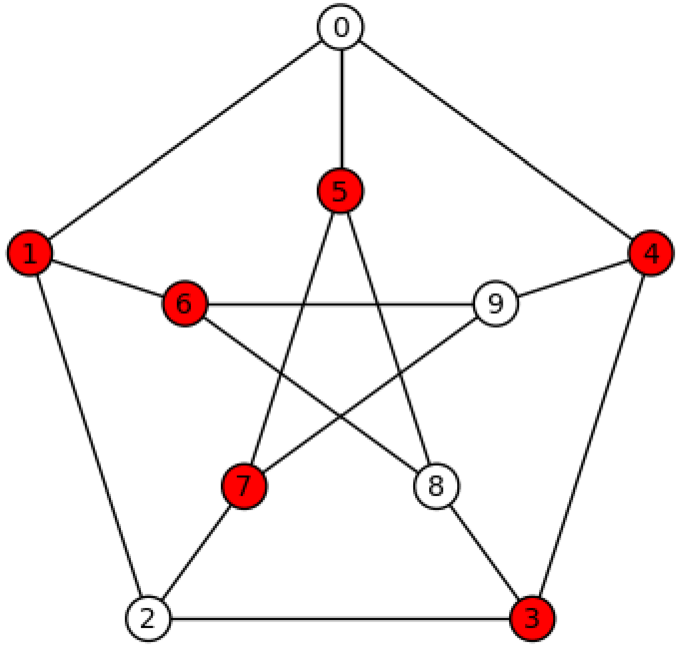
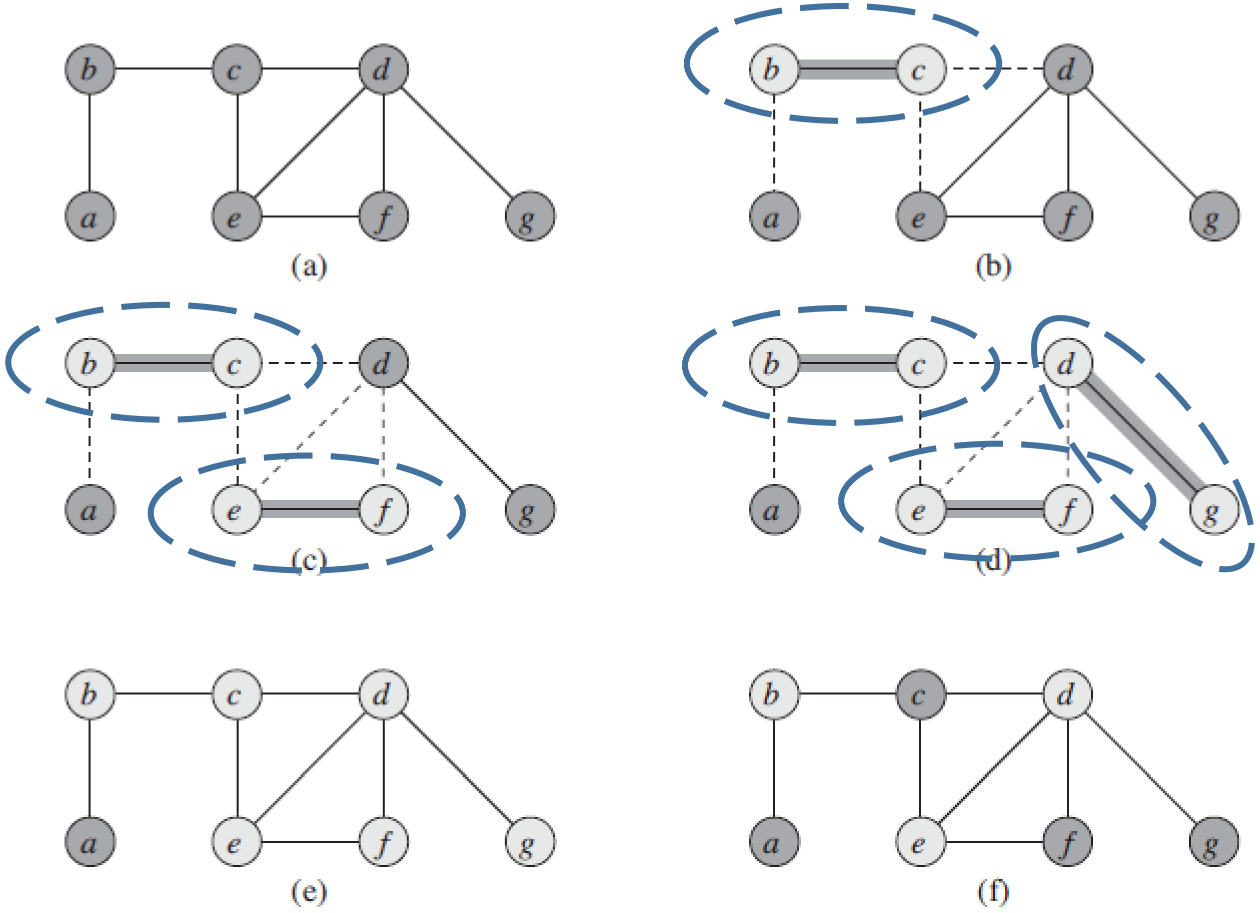


Approximate Algorithm for Vertex Cover Problem

APPROX-VERTEX-COVER(G)

```
C = ∅
E′ = G.E
while E′ ≠ ∅
      let (u, v) be an arbitrary edge of E′
      C = C ∪ {u, v}
      remove from E′ every edge incident on either u or v
return C
```



Vertex Cover Problem: Theorem

- **Theorem:**

AVC is a polynomial-time 2-approximation algorithm.

- **In other words:**

AVC produces a solution of cost C that is twice the optimal cost C*.

- **What is meant by cost in this problem?**

The size of the vertex cover set.

The smaller the size of the vertex cover set, the more optimal the solution.

Vertex Cover Problem: Theorem

- **Theorem:**

AVC is a polynomial-time 2-approximation algorithm.

- **To sum up, we need to proof that:**

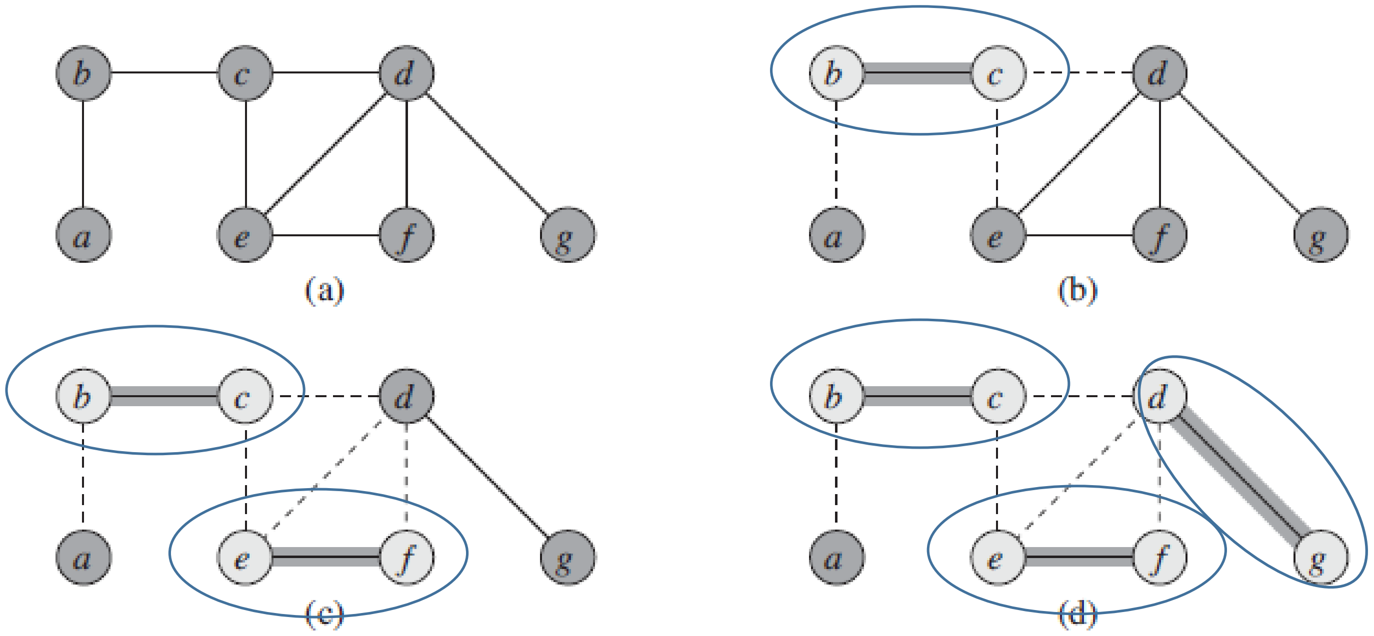
AVC produces a cover set that is TWICE the size of the optimal cover set.

Vertex Cover Problem: Theorem

- **Proof:**

1. Let A denote the set of edges picked at random by AVC (at line 4).
2. Any vertex cover must:
   - Cover the edges in A.
   - Include at least one endpoint of each edge in A.
3. No two edges in A share an endpoint.
4. No two edges in A are covered by the same vertex in C*

$\therefore |C^*| \geq A$



Vertex Cover Problem: Theorem

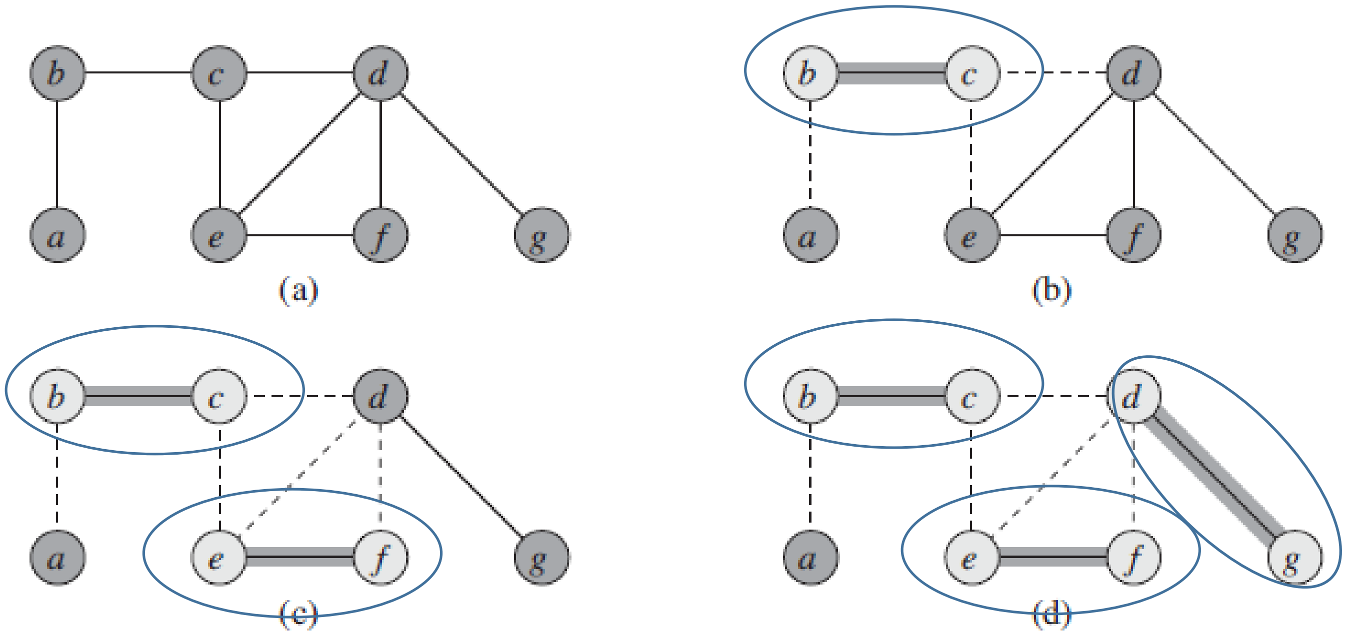
- **Proof:**

4. No two edges in A are covered by the same vertex in C*

   $\therefore |C^*| \geq |A|$

5. Each execution of line 4 picks an edge (in which neither of its endpoints is already in C), yielding an upper bound on the size of the vertex cover returned:

   $\therefore |C| = 2\ |A|$



Vertex Cover Problem: Theorem

- **Proof:**

From (4):

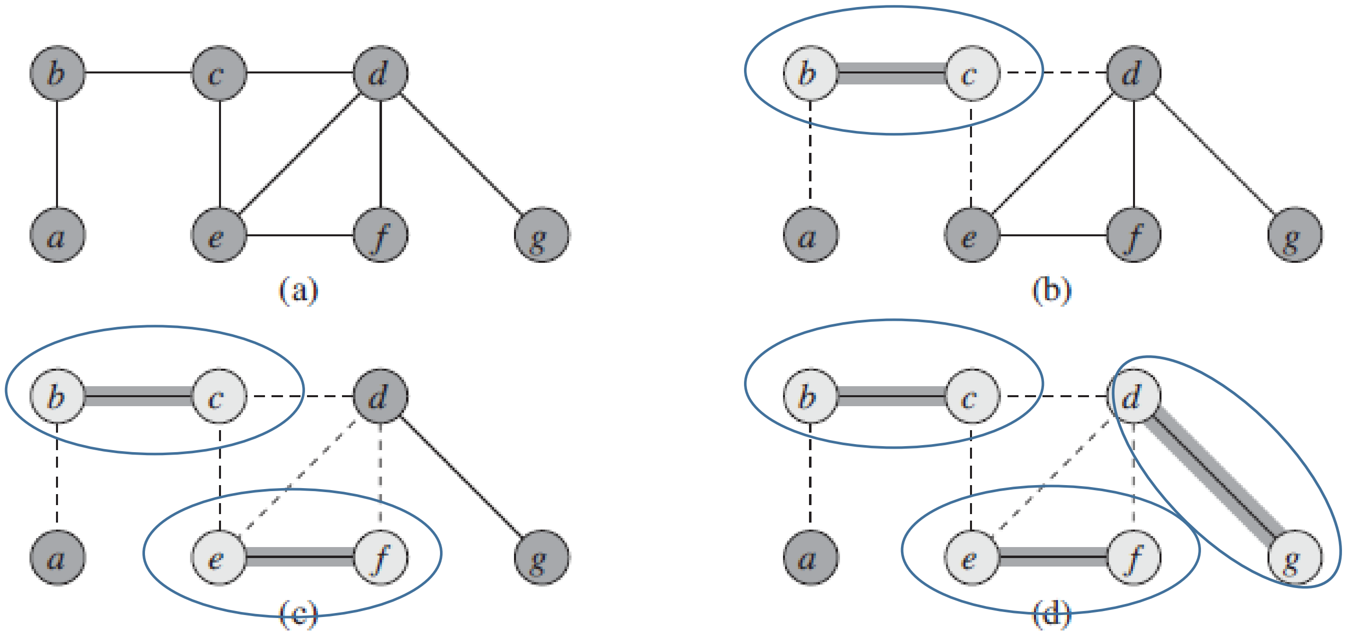
$\therefore |C^*| \geq |A|$ → (1)

From (5):

$\therefore |C| = 2\,|A|$ → (2)

Substituting (2) into (1)

$\therefore |C| \leq 2|C^*|$



Therfore, AVC is a 2-approximate algorithm.

THANK YOU