# Lagrange's Interpolating Polynomial for Approximating The Derivatives and The Calculation of The Electric Field and Charge from The Potential Function

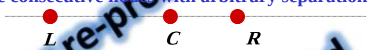




## Lagrange's Interpolating Polynomial for Approximating the Derivatives



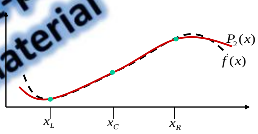
**Three consecutive nodes with arbitrary separation**




**Lagrange's interpolating polynomial is described as**

$$f(x) \approx P_2(x) = \frac{(x-x_C)(x-x_R)}{(x_L-x_C)(x_L-x_R)} f(x_L) + \frac{(x-x_L)(x-x_R)}{(x_C-x_L)(x_C-x_R)} f(x_C) + \frac{(x-x_L)(x-x_C)}{(x_R-x_L)(x_R-x_C)} f(x_R)$$

$P_2(x)$ is a second degree polynomial which coincides with the exact values of the function $f(x)$ at three nodes $L, R, C$, and approximates the function $f(x)$ in between these nodes.

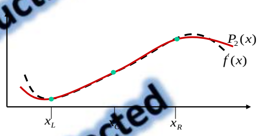





## Derivatives of Lagrange's Interpolating Polynomial



**Lagrange's interpolating polynomial is differentiated to obtain approximations for the first and second derivatives of a function based on known numerical values at three points of the function.**

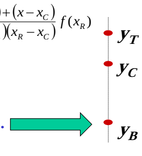



**In our case the left, center, and right values are sufficient to determine the first and second derivatives of a potential $V$ in $x$ direction at any point $x$ between the left and right positions $x_L$ and $x_R$.**

$$\frac{df(x)}{dx} = \frac{(x-x_C)+(x-x_R)}{(x_L-x_C)(x_L-x_R)} f(x_L) + \frac{(x-x_L)+(x-x_R)}{(x_C-x_L)(x_C-x_R)} f(x_C) + \frac{(x-x_L)+(x-x_C)}{(x_R-x_L)(x_R-x_C)} f(x_R)$$

$$\frac{d^2 f(x)}{dx^2} = \frac{2f(x_L)}{(x_L-x_C)(x_L-x_R)} + \frac{2f(x_C)}{(x_C-x_L)(x_C-x_R)} + \frac{2f(x_R)}{(x_R-x_L)(x_R-x_C)}$$

**Similar expressions can be obtained for the $y$ derivatives.**



## Quantities Computed From The Potential

- The electric field vectors $\vec{E} = -\nabla V$
- The charge on a conductor (Gauss's Law)
- The capacitance between conductors
- The characteristic impedance, effective dielectric constant, phase velocity.





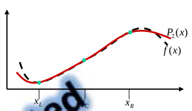
## Electric Field Distribution

After solving for the potential distribution, the electric field vector can be calculated at every node

$$\bar{E}(x,y) = -\nabla V(x,y) = -\frac{\partial V(x,y)}{\partial x}\hat{x} - \frac{\partial V(x,y)}{\partial y}\hat{y}$$

$$\bar{E}(x,y) = E_x(x,y)\hat{x} + E_y(x,y)\hat{y}$$

$$E_x = -\frac{\partial V(x,y)}{\partial x},\ E_y = -\frac{\partial V(x,y)}{\partial y}$$




Since we are using non-uniform grid, the potential can best be described at any arbitrary point using Lagrange's polynomials approximation

$$V(x,y) \approx P_2(x,y) = \frac{(x-x_C)(x-x_R)}{(x_L-x_C)(x_L-x_R)}V(x_L,y) + \frac{(x-x_L)(x-x_R)}{(x_C-x_L)(x_C-x_R)}V(x_C,y) + \frac{(x-x_L)(x-x_C)}{(x_R-x_L)(x_R-x_C)}V(x_R,y)$$

$$\frac{dV(x,y)}{dx} = \frac{2x-x_C-x_R}{(x_L-x_C)(x_L-x_R)}V(x_L,y) + \frac{2x-x_L-x_R}{(x_C-x_L)(x_C-x_R)}V(x_C,y) + \frac{2x-x_L-x_C}{(x_R-x_L)(x_R-x_C)}V(x_R,y)$$

$$E_x(x,y) = -\frac{2x-x_C-x_R}{(x_L-x_C)(x_L-x_R)}V_L\big|_y - \frac{2x-x_L-x_R}{(x_C-x_L)(x_C-x_R)}V_C\big|_y - \frac{2x-x_L-x_C}{(x_R-x_L)(x_R-x_C)}V_R\big|_y$$

$$E_y(x,y) = -\frac{2y-y_C-y_T}{(y_B-y_C)(y_B-y_T)}V_B\big|_x - \frac{2y-y_B-y_T}{(y_C-y_B)(y_C-y_T)}V_C\big|_x - \frac{2y-y_B-y_C}{(y_T-y_B)(y_T-y_C)}V_T\big|_x$$





## Electric Field Vectors

$$E_x(x,y) = -\frac{2x-x_C-x_R}{(x_L-x_C)(x_L-x_R)}V_L\big|_y - \frac{2x-x_L-x_R}{(x_C-x_L)(x_C-x_R)}V_C\big|_y - \frac{2x-x_L-x_C}{(x_R-x_L)(x_R-x_C)}V_R\big|_y$$

$$E_y(x,y) = -\frac{2y-y_C-y_T}{(y_B-y_C)(y_B-y_T)}V_B\big|_x - \frac{2y-y_B-y_T}{(y_C-y_B)(y_C-y_T)}V_C\big|_x - \frac{2y-y_B-y_C}{(y_T-y_B)(y_T-y_C)}V_T\big|_x$$

Thus in terms of the potential $V(i,j)$ and the coordinates $x(i,j)$ and $y(i,j)$ using the following definitions:

$$V_C = V_{(i,j)},\ V_T = V_{(i,j+1)},\ V_B = V_{(i,j-1)},\ V_R = V_{(i+1,j)},\ V_L = V_{(i-1,j)}$$
$$x_C = x(i,j),\ x_T = x(i,j+1),\ x_B = x(i,j-1),\ x_R = x(i+1,j),\ x_L = x(i-1,j)$$
$$y_C = y(i,j),\ y_T = y(i,j+1),\ y_B = y(i,j-1),\ y_R = y(i+1,j),\ y_L = y(i-1,j)$$

One can obtain the components of the electric field $E_x(x,y)$ and $E_y(x,y)$ at any arbitrary point $(x,y)$ in the computational domain.

## Electric Field Components at the Grid Points

$$E_x(x,y) = -\frac{2x - x_C - x_R}{(x_L - x_C)(x_L - x_R)} V_L\Big|_y - \frac{2x - x_L - x_R}{(x_C - x_L)(x_C - x_R)} V_C\,\Big|_y - \frac{2x - x_L - x_C}{(x_R - x_L)(x_R - x_C)} V_R\,\Big|_y$$

$$E_y(x,y) = -\frac{2y - y_C - y_T}{(y_B - y_C)(y_B - y_T)} V_L\,\Big|_x - \frac{2y - y_B - y_T}{(y_C - y_B)(y_C - y_T)} V_C\,\Big|_x - \frac{2y - y_B - y_C}{(y_T - y_B)(y_T - y_C)} V_T\,\Big|_x$$

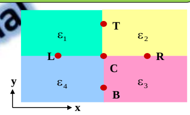
At $x = x_C \;, y = y_C$

$$E_x(x_c,y_c) = -\frac{x_C - x_R}{(x_L - x_C)(x_L - x_R)} V_L\,\Big|_{y_C} - \frac{(x_C - x_L) + (x_C - x_R)}{(x_C - x_L)(x_C - x_R)} V_C\,\Big|_{y_C} - \frac{x_C - x_L}{(x_R - x_L)(x_R - x_C)} V_R\,\Big|_{y_C}$$

$$E_y(x_c,y_c) = -\frac{y_C - y_T}{(y_B - y_C)(y_B - y_T)} V_B\,\Big|_{x_C} - \frac{(y_C - y_B) + (y_C - y_T)}{(y_C - y_B)(y_C - y_T)} V_C\,\Big|_{x_C} - \frac{y_C - y_B}{(y_T - y_B)(y_T - y_C)} V_T\,\Big|_{x_C}$$

For $\Delta x = \Delta y = h$

$$E_x(x_c,y_c) = \frac{V_L}{2h} - \frac{V_R}{2h} \;,\quad E_y(x_c,y_c) = \frac{V_B}{2h} - \frac{V_T}{2h}$$



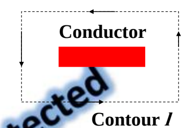


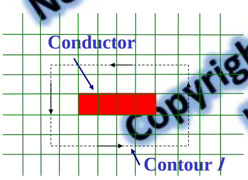


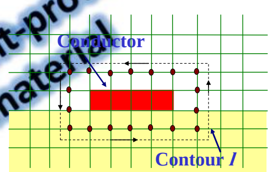

## Computation of the Total Charge on a Conductor

$$Q^{enc} = -\oint_A \varepsilon \frac{\partial V}{\partial n} \hat{a}_n \bullet \vec{d}A = -\oint_A \varepsilon \frac{\partial V}{\partial n} dA$$

$$\rho^{enc} = -\oint_l \varepsilon \frac{\partial V}{\partial n} dl$$







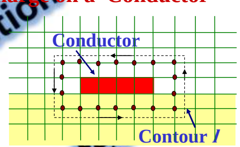

## Computation of The Total Charge on a Conductor

**Gauss's Law**

$$Q^{enc} = -\oint_A \varepsilon \frac{\partial V}{\partial n} \hat{a}_n \bullet \vec{d}A = -\oint_A \varepsilon \frac{\partial V}{\partial n} dA$$

$$\rho^{enc} = -\oint_l \varepsilon \frac{\partial V}{\partial n} dl$$

**Charge enclosed per unit length**




$$\rho^{enc} = \left[ -\int_{right} \varepsilon(y) \frac{\partial V}{\partial x} dy - \int_{top} \varepsilon(x) \frac{\partial V}{\partial y} dx + \int_{left} \varepsilon(y) \frac{\partial V}{\partial x} dy + \int_{bottom} \varepsilon(x) \frac{\partial V}{\partial y} dx \right]$$

with the approximations $\frac{\partial V}{\partial x} \simeq \frac{V_R - V_C}{x_R - x_C}$ and $\frac{\partial V}{\partial y} \simeq \frac{V_T - V_C}{y_T - y_C}$

$$\rho^{enc} = -\sum_{right} \frac{V_R - V_C}{x_R - x_C} \left[ \frac{\varepsilon_3 (y_C - y_B)/2 + \varepsilon_2 (y_T - y_C)/2}{dy} \right] dy - \sum_{top} \frac{V_T - V_C}{y_T - y_C} \left[ \frac{\varepsilon_1 (x_C - x_L)/2 + \varepsilon_2 (x_R - x_C)/2}{dx} \right] dx$$

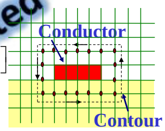
$$+ \sum_{left} \frac{V_C - V_L}{x_C - x_L} \left[ \frac{\varepsilon_1 (y_T - y_C)/2 + \varepsilon_4 (y_C - y_B)/2}{dy} \right] dy + \sum_{bottom} \frac{V_C - V_B}{y_C - y_B} \left[ \frac{\varepsilon_4 (x_C - x_L)/2 + \varepsilon_3 (x_R - x_C)/2}{dx} \right] dx$$

## Computation of The Total Charge on a Conductor

And by using coordinates indices $(i, j)$:

$$\rho^{enc} = -\sum_{right} \frac{V_R - V_C}{x_R - x_C}\left[\frac{\varepsilon_3(y_C - y_B) + \varepsilon_2(y_T - y_C)}{2}\right] - \sum_{top} \frac{V_T - V_C}{y_T - y_C}\left[\frac{\varepsilon_1(x_C - x_L) + \varepsilon_2(x_R - x_C)}{2}\right]$$
$$+\sum_{left} \frac{V_C - V_L}{x_C - x_L}\left[\frac{\varepsilon_1(y_T - y_C) + \varepsilon_4(y_C - y_B)}{2}\right] + \sum_{bottom} \frac{V_C - V_B}{y_C - y_B}\left[\frac{\varepsilon_4(x_C - x_L) + \varepsilon_3(x_R - x_C)}{2}\right]$$

$$\rho^{enc} = -\sum_{right} \frac{V_{(i+1,j)} - V_{(i,j)}}{x(i+1,j) - x(i,j)}\left[\frac{\varepsilon_{(i,j-1)}(y(i,j) - y(i,j-1)) + \varepsilon_{(i,j)}(y(i,j+1) - y(i,j))}{2}\right]$$
$$-\sum_{top} \frac{V_{(i,j+1)} - V_{(i,j)}}{y(i,j+1) - y(i,j)}\left[\frac{\varepsilon_{(i-1,j)}(x(i,j) - x(i-1,j)) + \varepsilon_{(i,j)}(x(i+1,j) - x(i,j))}{2}\right]$$
$$+\sum_{left} \frac{V_{(i,j)} - V_{(i-1,j)}}{x(i,j) - x(i-1,j)}\left[\frac{\varepsilon_{(i-1,j)}(y(i,j+1) - y(i,j)) + \varepsilon_{(i-1,j-1)}(y(i,j) - y(i,j-1))}{2}\right]$$
$$+\sum_{bottom} \frac{V_{(i,j)} - V_{(i,j-1)}}{y(i,j) - y(i,j-1)}\left[\frac{\varepsilon_{(i-1,j-1)}(x(i,j) - x(i-1,j)) + \varepsilon_{(i,j-1)}(x(i+1,j) - x(i,j))}{2}\right]$$







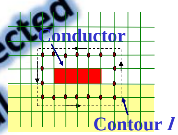

## Computation of The Total Charge on a Conductor

For $\Delta x = \Delta y = h$

$$\rho^{enc} = \left[-\int_{right} \varepsilon(y)\frac{\partial V}{\partial x}dy - \int_{top} \varepsilon(x)\frac{\partial V}{\partial y}dx + \int_{left} \varepsilon(y)\frac{\partial V}{\partial x}dy + \int_{bottom} \varepsilon(x)\frac{\partial V}{\partial y}dx\right]$$

$$\rho^{enc} = -\sum_{right}(V_R - V_C)\left[\frac{\varepsilon_3 + \varepsilon_2}{2}\right] - \sum_{top}(V_T - V_C)\left[\frac{\varepsilon_1 + \varepsilon_2}{2}\right]$$
$$+\sum_{left}(V_C - V_L)\left[\frac{\varepsilon_1 + \varepsilon_4}{2}\right] + \sum_{bottom}(V_C - V_B)\left[\frac{\varepsilon_4 + \varepsilon_3}{2}\right]$$

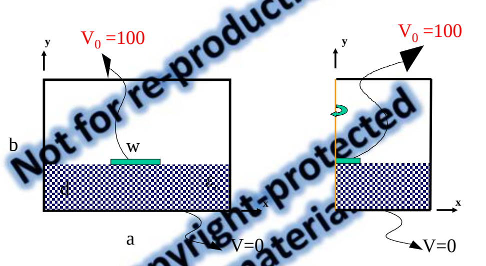
$$\rho^{enc} = -\sum_{right}\left(V_{(i+1,j)} - V_{(i,j)}\right)\left[\frac{\varepsilon_{(i,j-1)} + \varepsilon_{(i,j)}}{2}\right] - \sum_{top}\left(V_{(i,j+1)} - V_{(i,j)}\right)\left[\frac{\varepsilon_{(i-1,j)} + \varepsilon_{(i,j)}}{2}\right]$$
$$+\sum_{left}\left(V_{(i,j)} - V_{(i-1,j)}\right)\left[\frac{\varepsilon_{(i-1,j)} + \varepsilon_{(i-1,j-1)}}{2}\right] + \sum_{bottom}\left(V_{(i,j)} - V_{(i,j-1)}\right)\left[\frac{\varepsilon_{(i-1,j-1)} + \varepsilon_{(i,j-1)}}{2}\right]$$








## A Shielded Microstrip Line Geometry




Original problem    Computational domain

## Parameters of a Microstrip Line



**Capacitance**

$$C = \frac{\rho^{enc}}{V}$$

**Phase velocity**

$$v = \frac{c}{\sqrt{\varepsilon_{re}}} \quad , c \simeq 3x10^8$$

**Effective permittivity**

$$\varepsilon_{re} = \frac{\rho^{enc}}{\rho_0^{enc}} \;, \quad \rho_0^{enc} \text{ is the charge with } \varepsilon_r = 1$$

**Characteristic impedance**

$$Z_0 = \frac{1}{c\sqrt{CC_0}} \;, c \simeq 3x10^8, C_0 \text{ is the capacitance with } \varepsilon_r = 1$$



## Assignment



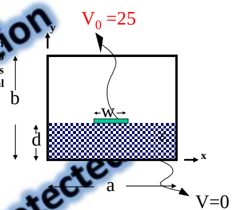
**For the shown cross-section of a shielded microstrip transmission line that extends along the z axis. Assuming that there is no variation along the z direction, calculate the following parameters of the microstrip line using the FD technique with central difference approximation, matrix inversion solution:**

**a) 3D graphs of potential distribution for**
**b) 2D graphs of electric field distribution for** $\varepsilon_r = 1$, and $\varepsilon_r = 12$
**c) total charge on the strip for** $\varepsilon_r = 1$, and $\varepsilon_r = 12$
**b) strip capacitance** $\varepsilon_r = 1$, and $\varepsilon_r = 12$
**e) strip characteristic impedance**
**f) strip effective permittivity**
**g) strip phase velocity.**

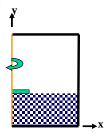
**The computations of the potential using the FD technique must be performed over one half of the cross-section of the geometry. Use Matlab functions to generate the complete potential distribution for the complete cross-section before computing the electric field, charge, etc.**

**Bonus points:**
**1- compute and sketch the charge distribution on the top surface of the strip.**

**2- use Lagrange's interpolation for the evaluation of the derivatives of the potential in the integrant part of Gauss's law.**




a = 7.5, b = 5.5, d = 1.5, w = 1.5, $\varepsilon_r = 12$
Thickness of the strip =0.05
All dimensions are in cm.






## End of Lecture