Finite Difference (FD) Approximations to Poisson's and Laplace's Equation and Truncation Errors

Siméon Denis Poisson
1781 - 1840

Pierre-Simon Laplace
1749 - 1827

Poisson's and Laplace's Equation

Electrostatic fields and capacitances due to a given distribution of a charge.

We will assume bounded media, but not necessarily homogeneous.

$$\nabla \bullet \vec{D} = \rho \quad \text{with } \vec{D} = \varepsilon \ \vec{E}$$

$$\nabla \bullet \vec{D} = \nabla \bullet \varepsilon \ \vec{E} = \varepsilon \ \nabla \bullet \vec{E} + \nabla \varepsilon \ \bullet \vec{E} = \rho$$

For homogeneous media, $\nabla \varepsilon = 0$

$$\nabla \bullet \vec{D} = \varepsilon \ \nabla \bullet \vec{E} = \rho \quad \text{and with } \vec{E} = -\nabla V$$

$$= \varepsilon \ \nabla \bullet \vec{E} = \varepsilon \ \nabla \bullet (-\nabla V) = -\varepsilon \nabla \bullet \nabla V = -\varepsilon \nabla^2 V = \rho$$

$$\nabla^2 V = {}^{-\rho}\!/\!_{\varepsilon} \quad \text{(Poisson's Equation)}$$

$$\nabla^2 V = 0 \quad \text{(Laplace's Equation)} \quad \text{when } \rho = 0,$$

Solution Methods

$$\nabla^2 V = {}^{-\rho}\!/\!_{\varepsilon} \quad \text{and} \quad \nabla^2 V = 0$$

- **Analytic closed form solution:**

Usually by the method of separation of variables and is possible for problems with boundaries that conforms with one of the coordinates system.

- **Numerical solution (with approximations):**

Using one of the well known numerical techniques (FD, MoM, MoL, TLM, …). This type of solution usually involves approximations to the geometry of the problem and/or approximation to the mathematical operators involved.

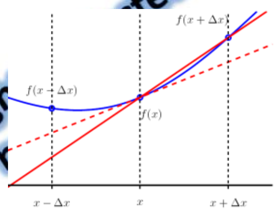
Poisson's and Laplace's equations require first and second order derivatives that can be approximated using the Finite Difference Methods.

Definition of Derivative: Forward Difference

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Forward difference formula

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

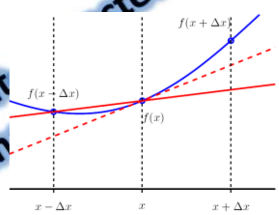


Definition of Derivative: Backward Difference

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x) - f(x-\Delta x)}{\Delta x}$$

Backward difference formula

$$f'(x) \approx \frac{f(x) - f(x-\Delta x)}{\Delta x}$$



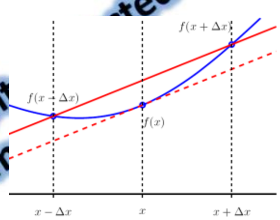
Definition of Derivative: Central Difference

Consider the average of the forward and the backward difference approximations for the derivative of the function *f(x)* at *x*

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad f'(x) \approx \frac{f(x) - f(x-\Delta x)}{\Delta x}$$

This average yields the central difference approximation at point x as

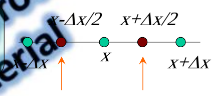
$$f'(x) \approx \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$$



First Derivative - Central Difference

$$f'(x) \approx \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$$

$$f'(x) \approx \frac{f(x+\Delta x/2) - f(x-\Delta x/2)}{\Delta x}$$

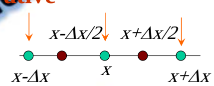


This central difference formulas are second order accurate.

The proof will be discussed later in the course.

Second Derivative

Using the central difference approximation for the first derivative



$$f''(x) = \frac{f'(x+\Delta x/2) - f'(x-\Delta x/2)}{\Delta x}$$

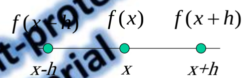
$$= \frac{\dfrac{f(x+\Delta x) - f(x)}{\Delta x} - \dfrac{f(x) - f(x-\Delta x)}{\Delta x}}{\Delta x}$$

$$f''(x) = \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{\Delta x^2}$$

Taylor Series Approach for Approximating the Derivatives

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + \frac{h^4}{24} f''''(x) \cdots$$

An exact representation of $f(x+h)$ if infinite number of terms are used.



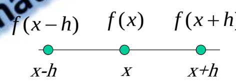
Truncation Error - Forward Difference

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2}f''(x)+\frac{h^3}{6}f'''(x)+...,$$

or, $$f'(x)=\frac{f(x+h)-f(x)}{h}-\frac{h}{2}f''(x)-\frac{h^2}{6}f'''(x)-...,$$

usually written as $$f'(x)=\frac{f(x+h)-f(x)}{h}+O(h)$$

Hence the forward difference formula is first-order accurate.



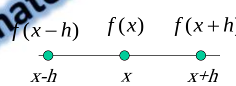
Truncation Error- Backward Difference

$$f(x-h)=f(x)-hf'(x)+\frac{h^2}{2}f''(x)-\frac{h^3}{6}f'''(x)+...,$$

or, $$f'(x)=\frac{f(x)-f(x-h)}{h}+\frac{h}{2}f''(x)-\frac{h^2}{6}f'''(x)-...,$$

usually written as $$f'(x)=\frac{f(x)-f(x-h)}{h}+O(h)$$

Hence the backward difference formula is first-order accurate.



First Derivative Using Taylor Series

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2}f''(x)+\frac{h^3}{6}f'''(x)+...,$$

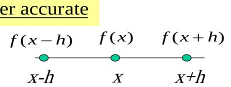
$$f(x-h)=f(x)-hf'(x)+\frac{h^2}{2}f''(x)-\frac{h^3}{6}f'''(x)+...,$$

taking the difference gives

$$f(x+h)-f(x-h)=2hf'(x)+\frac{2h^3}{6}f'''(x)$$

or $$f'(x)=\frac{f(x+h)-f(x-h)}{2h}+O(h^2)$$

Hence the first derivative based on the central difference formula is second-order accurate



Second Derivative Using Taylor Series

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f''''(x) + ...,$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f''''(x) - ...,$$

and summing gives

$$f(x+h) + f(x-h) = 2f(x) + 2\frac{h^2}{2}f''(x) + 2\frac{h^4}{24}f''''(x) + O(h^6)$$

or $f''(x) = \dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2)$

Hence the second derivative based on the central difference formula is second-order accurate

$f(x-h)$ $f(x)$ $f(x+h)$

x-h x x+h

Finite Difference Approximations

Derivative	Finite Difference Approximation	Type	Error
$\frac{\partial f}{\partial x}$	$\frac{f_{i+1} - f_i}{\Delta x}$	FD	$O(\Delta x)$
	$\frac{f_i - f_{i-1}}{\Delta x}$	BD	$O(\Delta x)$
	$\frac{f_{i+1} - f_{i-1}}{2\Delta x}$	CD	$O(\Delta x)^2$
	$\frac{-f_{i+2} + 4f_{i+1} - 3f_i}{2\Delta x}$	FD	$O(\Delta x)^2$
	$\frac{3f_i - 4f_{i-1} + f_{i-2}}{2\Delta x}$	BD	$O(\Delta x)^2$
	$\frac{-f_{i+2} + 8f_{i+1} - 8f_{i-1} + f_{i-2}}{12\Delta x}$	CD	$O(\Delta x)^4$

Derivative	Finite Difference Approximation	Type	Error
$\frac{\partial^2 f}{\partial x^2}$	$\frac{f_{i+2} - 2f_{i+1} + f_i}{(\Delta x)^2}$	FD	$O(\Delta x)$
	$\frac{f_i - 2f_{i-1} + f_{i-2}}{(\Delta x)^2}$	BD	$O(\Delta x)$
	$\frac{f_{i+1} - 2f_i + f_{i-1}}{(\Delta x)^2}$	CD	$O(\Delta x)^2$
	$\frac{-f_{i+2} + 16f_{i+1} - 30f_i + 16f_{i-1} - f_{i-2}}{12(\Delta x)^2}$	CD	$O(\Delta x)^4$

Where
FD= Forward Difference
BD= Backward Difference
CD= Central Difference

f_{i-2} f_{i-1} f_i f_{i+1} f_{i+2}

$x-2\Delta x$ $x-\Delta x$ x $x+\Delta x$ $x+2\Delta x$

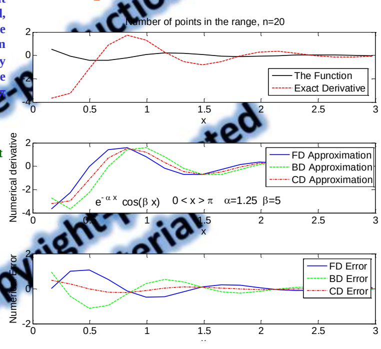
Example # 2

Develop a Matlab program that compute the forward, backward, and central differences for the first derivative of the function $y = e^{-\alpha x}\cos(\beta x)$ at equally spaced points within the range of x determined by 0 <= x <= π and with α = 1.25 and β = 5.

Sketch the function and its exact derivative in one figure. In another figure sketch the derivatives based on the approximations BD, CD, and FD.

In a separate figure sketch the errors associated with the approximate numerical derivatives.

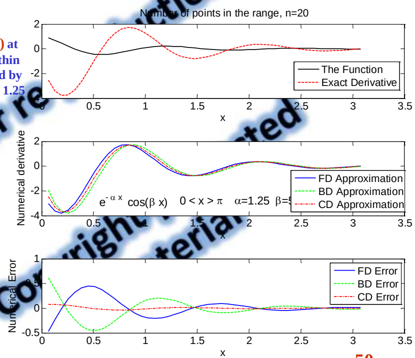
All figures should be for the entire range of x.



n = 20

Example # 2 with n=50

The derivative of the function $y= e^{-\alpha x} \cos(\beta x)$ at equally spaced points within the range of x determined by $0 <= x <= \pi$ and with $\alpha = 1.25$ and $\beta = 5$.



n = 50

Matlab Program for Example # 2

```
% Example # 2
% By Dr. Atef Elsherbeni
% Last update August 27, 2007

clear all; clc;
setfigures  % defines default parameters for figures

alpha= 1.25;  beta = 5;  n = 20;
x=linspace(0,pi,n); dx=x(2)-x(1);

f=exp(-alpha*x).*cos(beta*x);          % The function at the n points
fp=exp(-alpha*x).*(-beta*sin(beta*x)-alpha*cos(beta*x)); % the Derivative of the function

fd(2:n-1)= (f(3:n)-f(2:n-1))/dx;     % Forward difference
bd(2:n-1)= (f(2:n-1)-f(1:n-2))/dx;     % Backward differen
cd(2:n-1)= (f(3:n)-f(1:n-2))/(2*dx);     % Central difference

fde(2:n-1)=(fd(2:n-1)-fp(2:n-1));
bde(2:n-1)=(bd(2:n-1)-fp(2:n-1));
cde(2:n-1)=(cd(2:n-1)-fp(2:n-1));

subplot(3,1,1)
plot(x(2:n-1),f(2:n-1),'k-',x(2:n-1),fp(2:n-1),'r--' );
legend('The Function', 'Exact Derivative'); xlabel('x');
title('Number of points in the range, n=20');

subplot(3,1,2)
plot(x(2:n-1),fd(2:n-1),'b-',x(2:n-1),bd(2:n-1),'g--',x(2:n-1),cd(2:n-1),'r-.' );
legend('FD Approximation', 'BD Approximation', 'CD Approximation');
ylabel('Numerical derivative'); xlabel('x');

subplot(3,1,3)
plot(x(2:n-1),fde(2:n-1),'b-',x(2:n-1),bde(2:n-1),'g--',x(2:n-1),cde(2:n-1),'r-.' );
legend('FD Error', 'BD Error', 'CD Error');
ylabel('Numerical Error'); xlabel('x');
%
text(1.2,4.2,' 0 < x > \pi    \alpha=1.25  \beta=5','fontsize', 16);
text(0.19,5.1,'  e^{- \alpha x } cos(\beta x)','fontsize', 16);
```


Homework # 2

A) **Verify analytically that the shown expressions are correct.**

$$\frac{\partial^2 f}{\partial x^2} \cong \frac{f_{i+2}-2f_{i+1}+f_i}{(\Delta x)^2} \quad \text{FD} \quad O(\Delta x)$$

$$\cong \frac{f_i-2f_{i-1}+f_{i-2}}{(\Delta x)^2} \quad \text{BD} \quad O(\Delta x)$$

$$\cong \frac{f_{i+1}-2f_i+f_{i-1}}{(\Delta x)^2} \quad \text{CD} \quad O(\Delta x)^2$$

$$\cong \frac{-f_{i+2}+16f_{i+1}-30f_i+16f_{i-1}-f_{i-2}}{12(\Delta x)^2} \quad \text{CD} \quad O(\Delta x)^4$$

Homework # 2

B) Develop an m-file that compute the forward-, backward-, and central- difference formulas for the first derivative of the analytical function $\exp(-\alpha x)\sin(\beta x)$ with $\alpha = 0.3$, $\beta=1.2$ at equally spaced N points within an the interval $0 <= x <= 6\pi$.

Notice that the forward-difference formula cannot be applied at $x = 6\pi$, and the backward-difference formula cannot be applied at $x = 0$.

In one figure, let the program sketch the errors based on these three approximations for N = 31.

Repeat for the case when N = 61.

Comment on the two figures.

End of Lecture # 2