Introduction to Computational Electromagnetics
Lecture 2: Using the MoM!

ELC 657 – Spring 2017

Department of Electronics and Communications Engineering

Faculty of Engineering – Cairo University

Outline

1. 2D PEC Cylinder – Electrostatic Case
   - Formulation
   - Example: 2D Flat Strip
   - Convergence Studies and Parameter Extraction

2. Testing
   - Point Matching
   - Other Testing Functions

3. Background Theory
   - The Projection Theorem
   - The MoM Procedure

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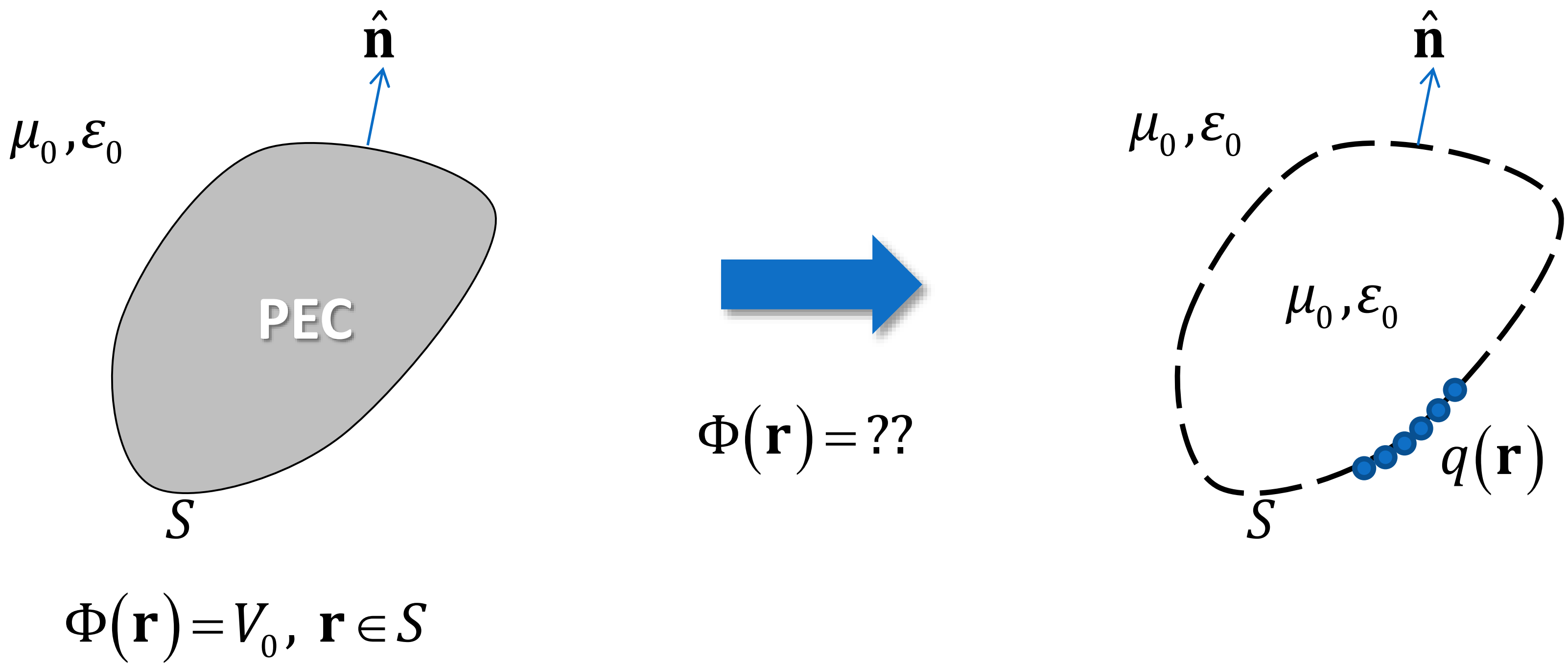
Formulation

Equivalent Problem



It is required to find the electric potential everywhere in space due to a PEC (cylinder) charged to a constant potential V_0.

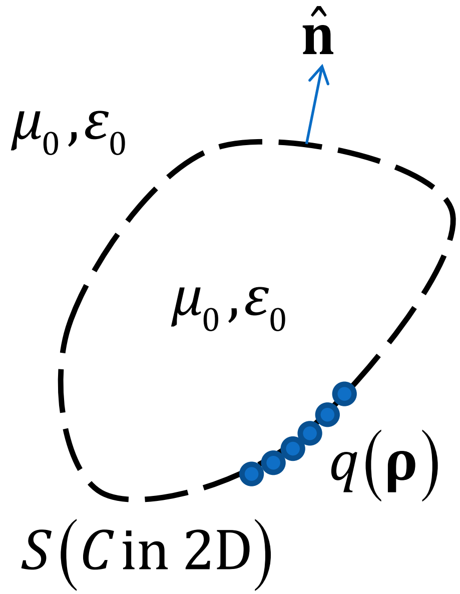
Formulation

Potential In Terms of Charge Distribution

$$\Phi(\boldsymbol{\rho}) = \frac{1}{\varepsilon_0}\int_C q(\boldsymbol{\rho}')\lim_{h\to\infty}\int_{-h}^{h}\frac{1}{4\pi R}dz'dl'$$

$$R = \sqrt{z'^2 + |\boldsymbol{\rho}-\boldsymbol{\rho}'|^2}$$

$$\Phi(\boldsymbol{\rho}) = \frac{1}{2\pi\varepsilon_0}\int_C q(\boldsymbol{\rho}')\ln\left|\frac{1}{\boldsymbol{\rho}-\boldsymbol{\rho}'}\right|dl'$$



If the charge distribution q is known, then the potential can be computed. But we don't know q! What do we know about it?

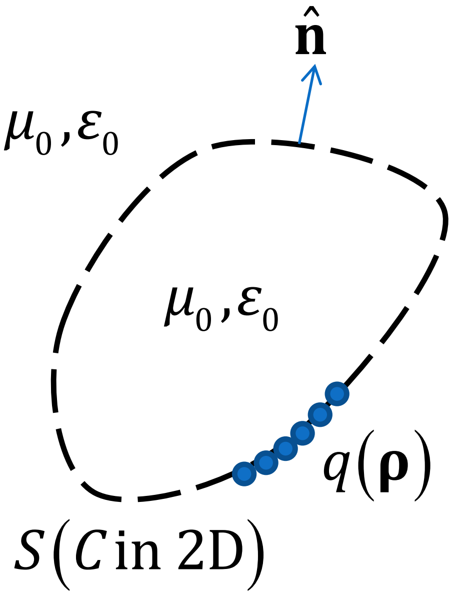
Formulation

Operator Equation Formulation

$$\Phi(\boldsymbol{\rho}) = V_0, \ \boldsymbol{\rho} \in C$$

$$\frac{-1}{2\pi\varepsilon_0} \int_C q(\boldsymbol{\rho}') \ln|\boldsymbol{\rho} - \boldsymbol{\rho}'| \, dl' = V_0, \ \boldsymbol{\rho} \in C$$

$$L\{q(\boldsymbol{\rho})\} = V_0, \ \boldsymbol{\rho} \in C$$



The above operator (integral) equation should be solved to determine the unknown charge distribution.

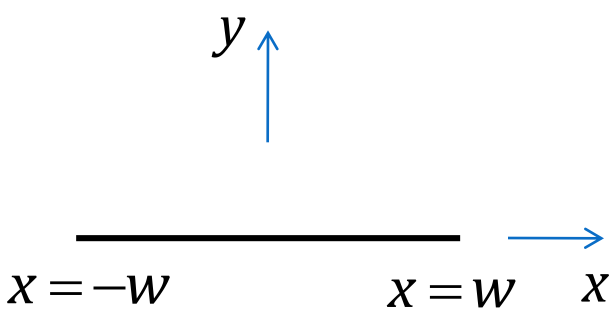
Example: 2D Flat Strip

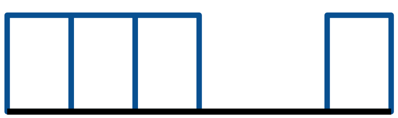
Basis Function Expansion

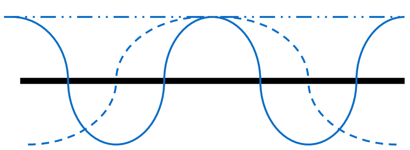
$$\frac{-1}{2\pi\varepsilon_0}\int_{-w}^{w} q(x')\ln|x-x'|\,dx' = V_0,\ x\in(-w,w)$$

$$q(x)\approx\sum_{n=1}^{N} q_n f_n(x)$$

Basis Functions







How to choose the basis functions?
Do we know anything about the charge behavior in this case?

Example: 2D Flat Strip

Geometry and Charge Discretization

$$\frac{-1}{2\pi\varepsilon_0}\sum_{n=1}^{N} q_n \int_{-w}^{w} \Pi_n(x')\ln|x-x'|\,dx' \approx V_0,\ x\in(-w,w)$$

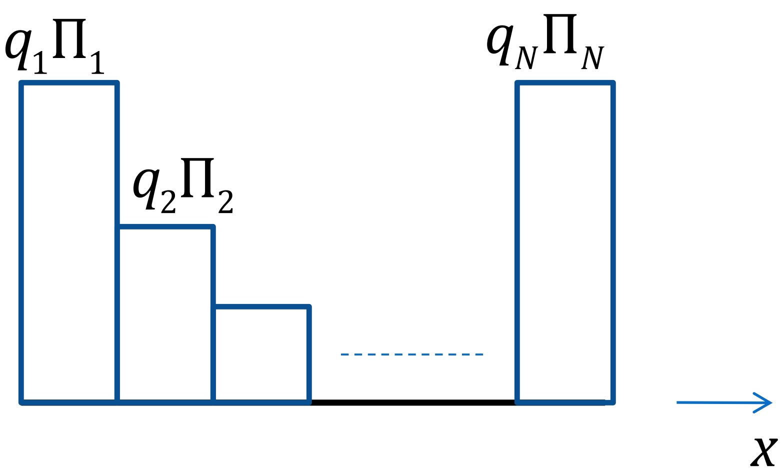
Geometry Discretization

$$\Pi_n(x)=\begin{cases}1, & x_{n-1}<x<x_n\\ 0, & \text{otherwise}\end{cases}$$

Basis (Expansion) Functions

$$x_n = -w + n\Delta$$

$$\Delta = \frac{2w}{N}$$



$$q_n = ??$$

Example: 2D Flat Strip

Point Matching

$$\frac{-1}{2\pi\varepsilon_0}\sum_{n=1}^{N} q_n \int_{x_{n-1}}^{x_n} \ln\left|x - x'\right| dx' \approx V_0,\ x \in (-w, w)$$

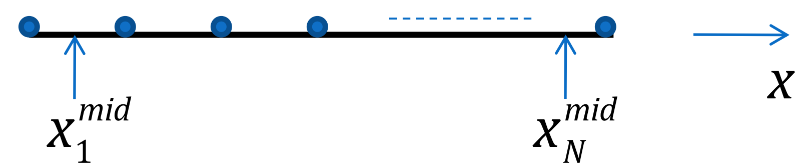
Point Matching

$$\boxed{\frac{-1}{2\pi\varepsilon_0}\sum_{n=1}^{N} q_n \int_{x_{n-1}}^{x_n} \ln\left|x_m^{mid} - x'\right| dx' = V_0,\ m = 1,2,\ldots,N}$$

$$x_m^{mid} = \frac{x_m + x_{m-1}}{2}$$

$$\sum_{n=1}^{N} q_n s_{mn} = V_0$$

$$s_{mn} = \frac{-1}{2\pi\varepsilon_0}\int_{x_{n-1}}^{x_n} \ln\left|x_m^{mid} - x'\right| dx'$$



What happens in between the matching points?
Can the integrals be evaluated analytically?
What is the physical interpretation of s_{mn}?

Example: 2D Flat Strip

Assignment #1

Find a closed-form expression for the integral below and verify the obtained expression for different values of the integral parameters.

$$\int_a^b \ln\left|x - x'\right| dx'$$

Example: 2D Flat Strip

MoM Matrix Equation

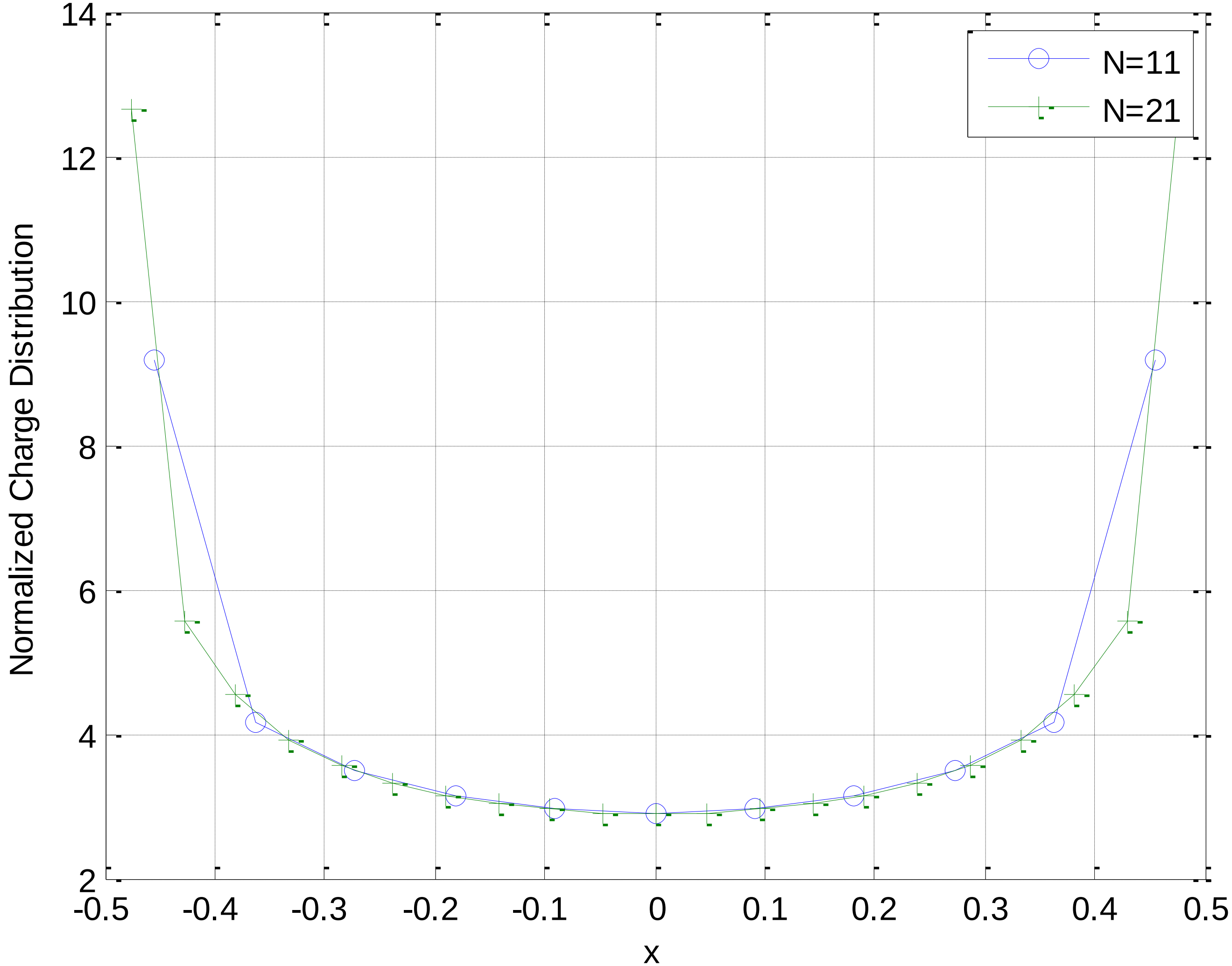
$$\sum_{n=1}^{N} q_n s_{mn} = V_0, \quad m,n = 1,2,\ldots,N$$

$$\underline{\underline{\mathbf{S}}}\,\underline{\mathbf{q}} = \underline{\mathbf{v}} \quad \Rightarrow \qquad \underline{\mathbf{q}} = \underline{\underline{\mathbf{S}}}^{-1}\underline{\mathbf{v}}$$

What matrix inversion procedure should be used? Direct, Gauss elimination, LU decomposition, iterative …?
Where is the point of reference potential?
Is the obtained solution unique? What are the uniqueness conditions for electrostatics?
What is the problem of pulse expansion (compared to triangle basis)?
What quantities can be computed/plotted once the charge distribution is determined?

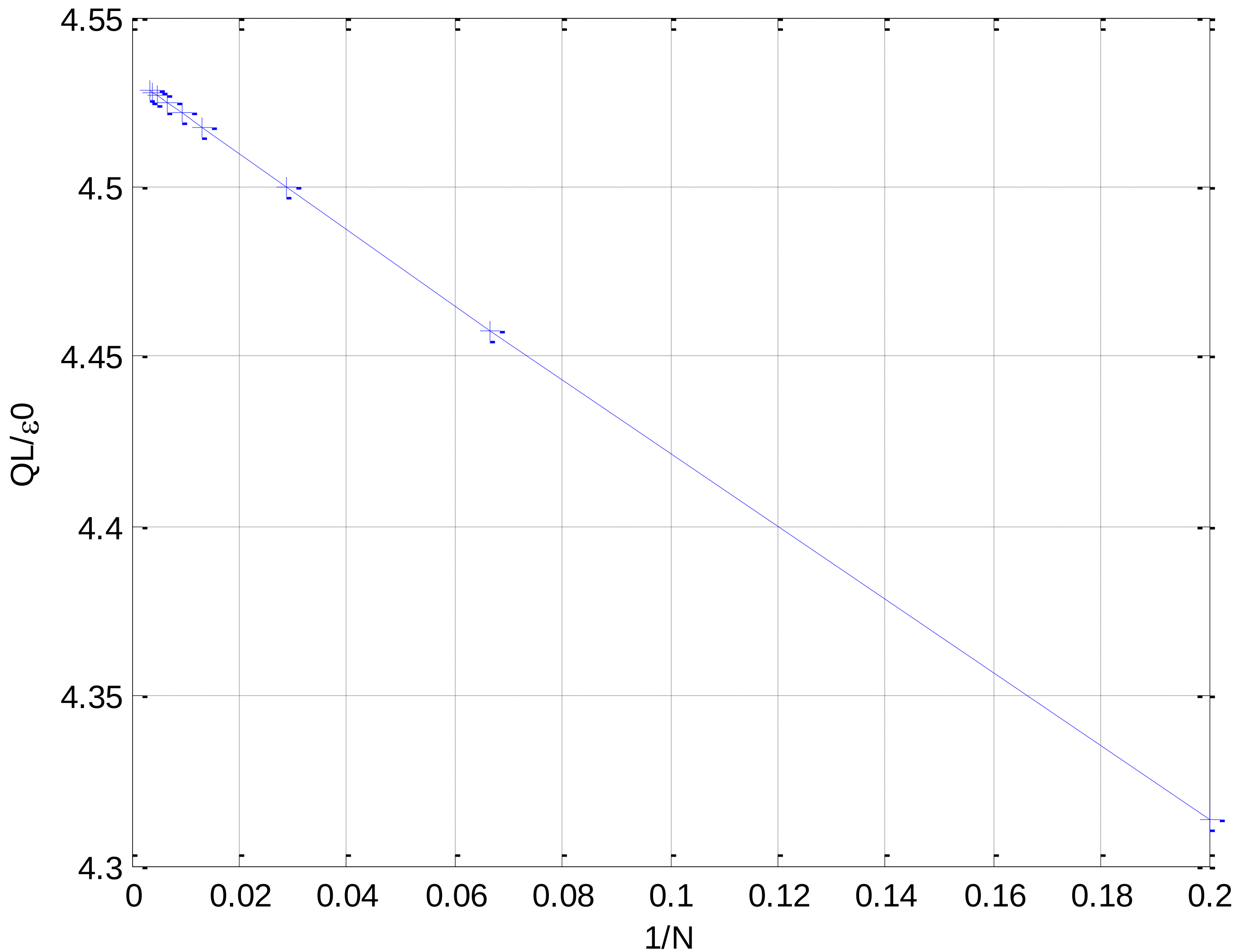
Convergence Studies and Parameter Extraction

Convergence of Charge Distribution



Convergence Studies and Parameter Extraction

Convergence of Total Charge



Convergence Studies and Parameter Extraction

Assignment #2

Write a Matlab code to determine the linear charge distribution on a PEC strip of width 1m and charged to a potential of 1V. Then:

1. Plot the charge distribution on the strip for increasing number of unknowns?
2. Study the convergence of the total charge with the number of unknowns and estimate the asymptotic total charge.
3. Plot the error in the potential distribution on the strip.
4. Plot the potential distribution in the region surrounding the strip.
5. Plot the electric field distribution in the same region.
6. Bonus: determine the location of the reference (zero-potential) point.
7. Bonus: use the symmetry to reduce the number of unknowns in your formulation.

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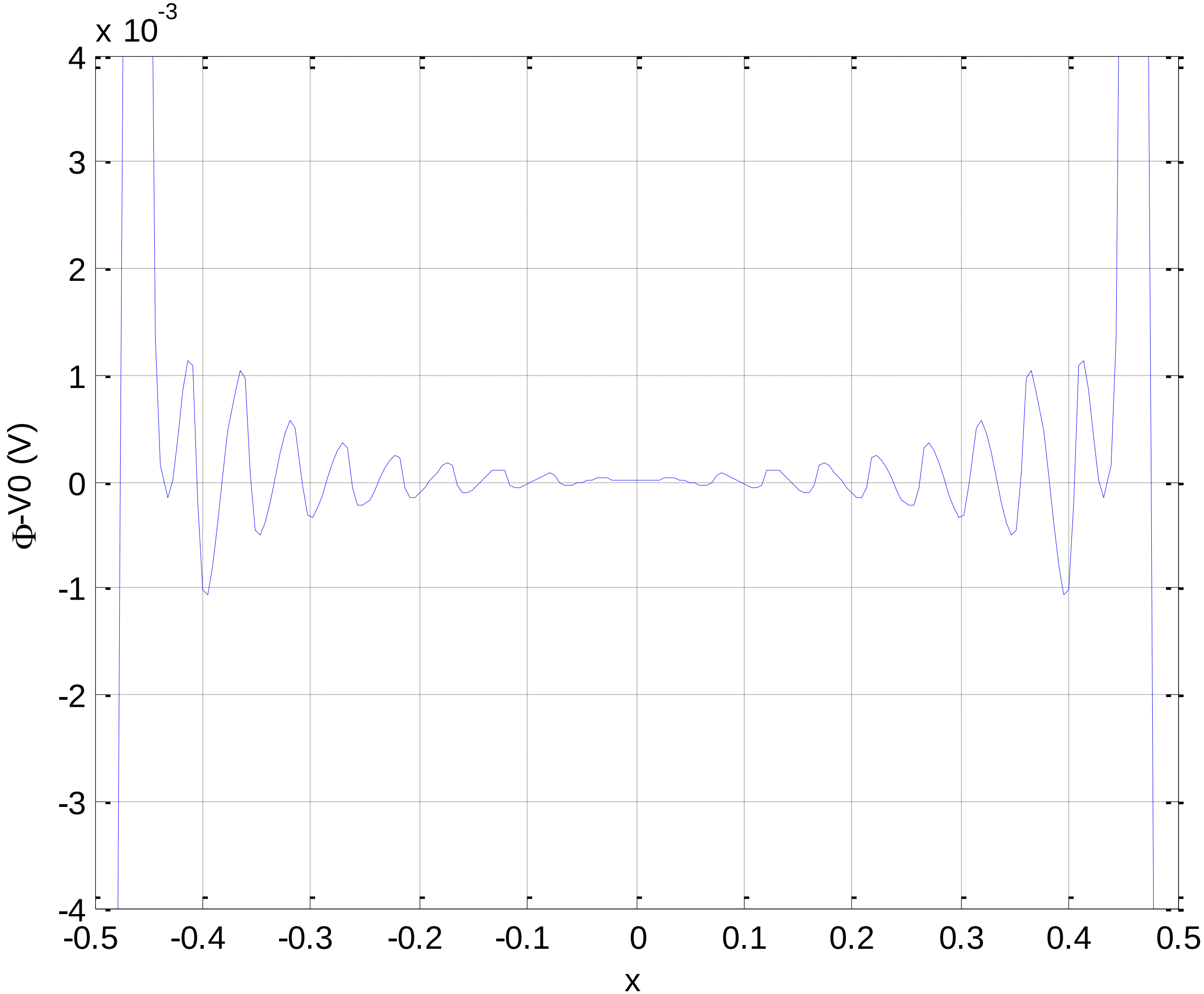
Point Matching

Error in Potential



Other Testing Functions

Pulse Testing

Point matching results in potential distribution satisfying the boundary conditions at the matching points only.
The distribution deviates from the correct value in between these points.
It might be better to require that the boundary conditions be satisfied in an averaged sense rather than at specific points.

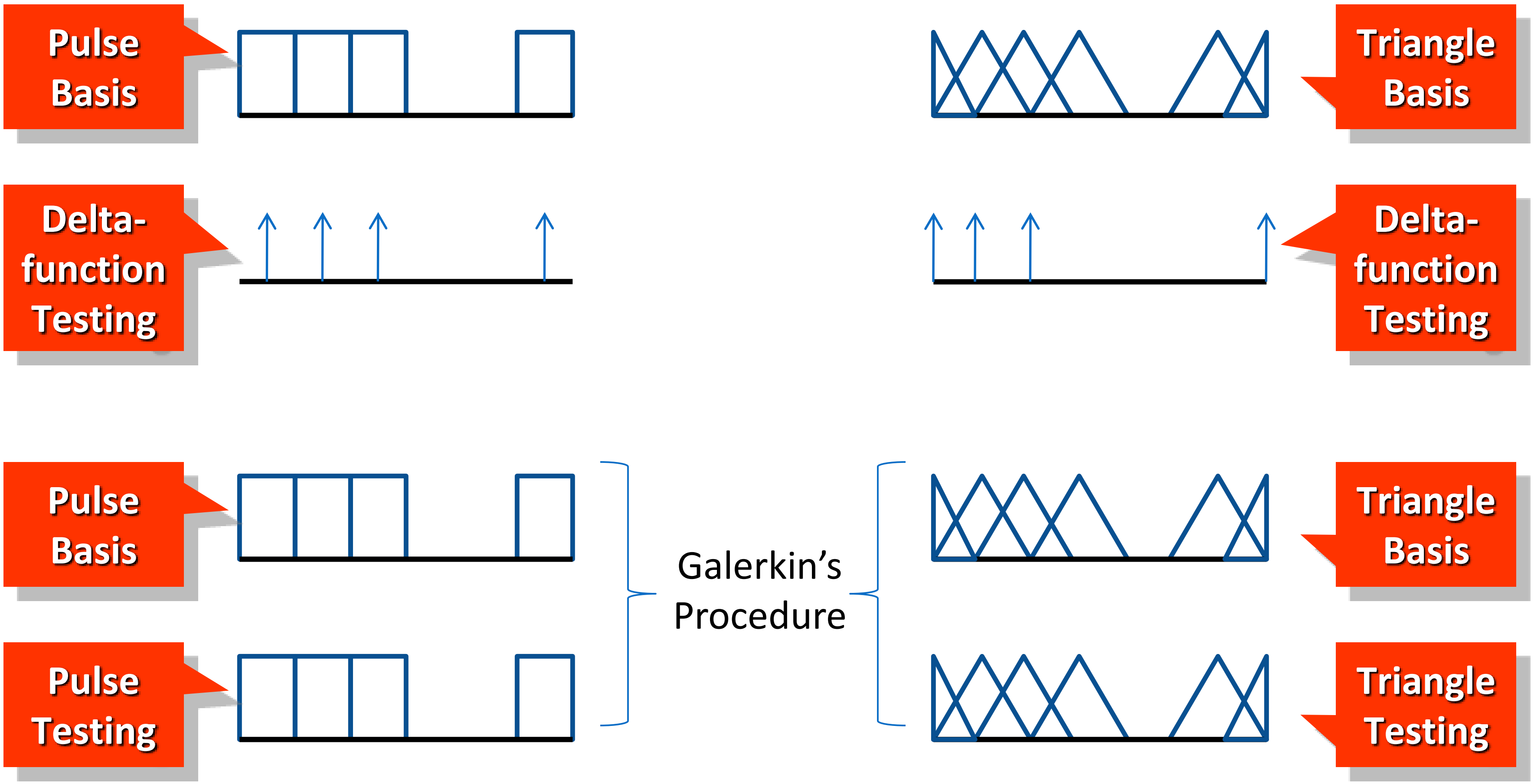
$$\frac{-1}{2\pi\varepsilon_0}\sum_{n=1}^{N} q_n \int_{x_{n-1}}^{x_n} \ln|x-x'|\,dx' \approx V_0,\; x\in(-w,w)$$

$$\int_{x_{m-1}}^{x_m} \Pi_m(x)\frac{-1}{2\pi\varepsilon_0}\sum_{n=1}^{N} q_n \int_{x_{n-1}}^{x_n} \ln|x-x'|\,dx'dx = \int_{x_{m-1}}^{x_m} \Pi_m(x)V_0 dx,\; x\in(-w,w)$$

$$\frac{-1}{2\pi\varepsilon_0\Delta}\sum_{n=1}^{N} q_n \int_{x_{m-1}}^{x_m}\int_{x_{n-1}}^{x_n} \ln|x-x'|\,dx'dx = V_0,\; x\in(-w,w)$$

Other Testing Functions

Different Possibilities



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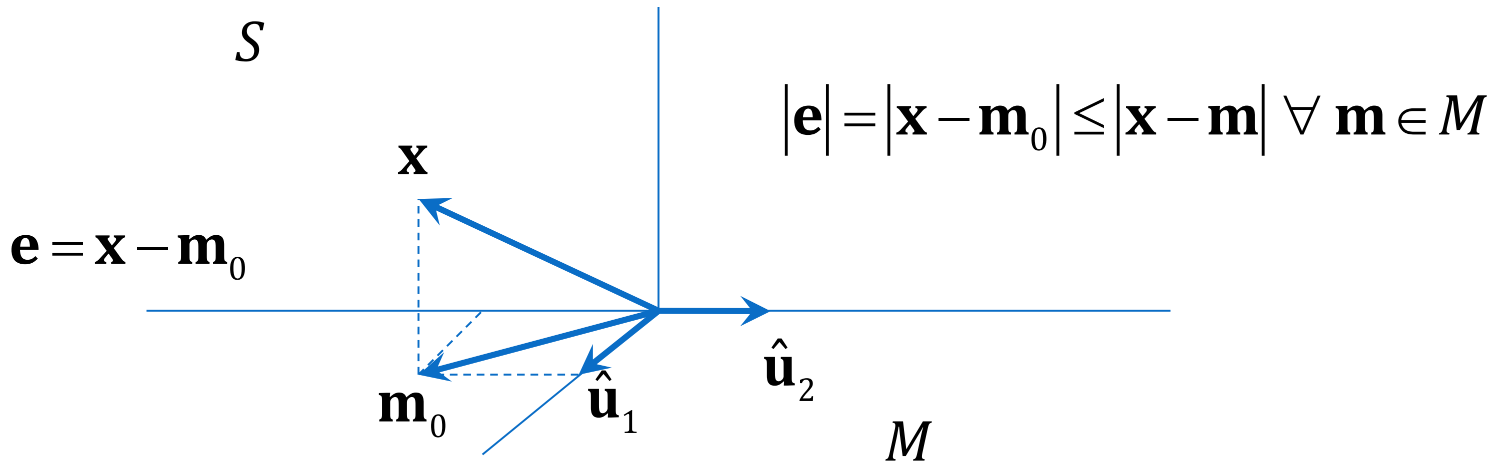
The Projection Theorem

Theorem Statement



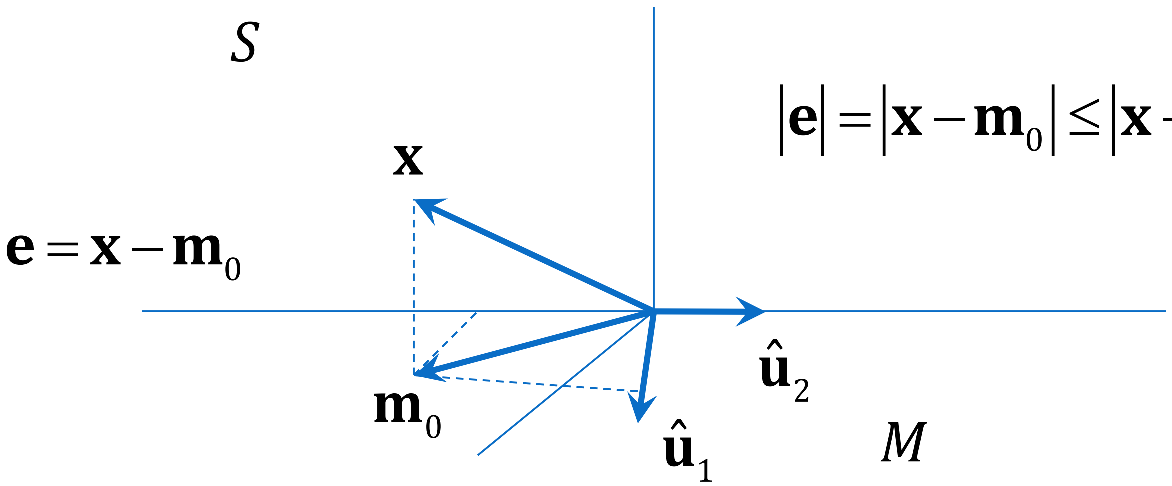
For any vector **x** in a Hilbert space *H*, there exists a unique vector $\mathbf{m}_0$ element of the sub-space *M* of *H*, such that :

$$|\mathbf{x} - \mathbf{m}_0| \le |\mathbf{x} - \mathbf{m}| \ \forall \ \mathbf{m} \in M$$

Furthermore, a necessary and sufficient condition that $\mathbf{m}_0$ be the unique minimizing vector is that $\mathbf{x}-\mathbf{m}_0$ be orthogonal to *M*.

The Projection Theorem

Theorem Applications



$$|\mathbf{e}| = |\mathbf{x} - \mathbf{m}_0| \leq |\mathbf{x} - \mathbf{m}| \ \forall \ \mathbf{m} \in M$$

The projection theorem can be used to find "best approximations" in terms of linearly independent, but non-orthonormal basis vectors.

The Projection Theorem

Best Approximation

$$\mathbf{x} = \begin{pmatrix} x_1 & x_2 & \cdots & x_N \end{pmatrix} \in S \qquad \{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_M\}$$

$$\mathbf{x} \cong \sum_{m=1}^{M} c_m \mathbf{y}_m \equiv \hat{\mathbf{x}}$$

Projection Theorem

$$\langle \mathbf{x} - \hat{\mathbf{x}}, \mathbf{y}_k \rangle = 0, \quad k = 1, 2, \cdots, M$$

$$\sum_{m=1}^{M} c_m \langle \mathbf{y}_k, \mathbf{y}_m \rangle = \langle \mathbf{y}_k, \mathbf{x} \rangle, \quad k = 1, 2, \cdots, M$$

This is a system of simultaneous equations, which can be cast in matrix form and solved for the unknowns.

The Projection Theorem

Best Approximation

$$\begin{bmatrix} \langle \mathbf{y}_1, \mathbf{y}_1 \rangle & \langle \mathbf{y}_1, \mathbf{y}_2 \rangle & \cdots & \langle \mathbf{y}_1, \mathbf{y}_M \rangle \\ \langle \mathbf{y}_2, \mathbf{y}_1 \rangle & \langle \mathbf{y}_2, \mathbf{y}_2 \rangle & & \\ \vdots & & \ddots & \vdots \\ \langle \mathbf{y}_M, \mathbf{y}_1 \rangle & \langle \mathbf{y}_M, \mathbf{y}_2 \rangle & \cdots & \langle \mathbf{y}_M, \mathbf{y}_M \rangle \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_M \end{bmatrix} = \begin{bmatrix} \langle \mathbf{y}_1, \mathbf{x} \rangle \\ \langle \mathbf{y}_2, \mathbf{x} \rangle \\ \vdots \\ \langle \mathbf{y}_M, \mathbf{x} \rangle \end{bmatrix}$$

What are the special cases for the matrix above?

The Projection Theorem

Example

Determine the best approximation of **x** in the subspace defined by the set {$\mathbf{y}_n$: n =1,2,3} for the following cases. Comment.

$$\mathbf{x} = \begin{pmatrix} 4 & 2 & -1 & 2 \end{pmatrix}$$

$$\mathbf{y}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$
$$\mathbf{y}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$
$$\mathbf{y}_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{x} = \begin{pmatrix} 4 & 2 & -1 & 2 \end{pmatrix}$$

$$\mathbf{y}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$
$$\mathbf{y}_2 = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}$$
$$\mathbf{y}_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

The MoM Procedure

Linear Operator: Definition

$$\mathbf{f} = L\{\mathbf{x}\}$$

$$L\{a_1\mathbf{x}_1 + a_2\mathbf{x}_2\} = a_1 L\{\mathbf{x}_1\} + a_2 L\{\mathbf{x}_2\}$$

An operator *L* is a mapping that assigns to a vector **x** in *S* another vector *L*{**x**} in *S*. The domain of the operator is the set of vectors for which the mapping is defined. The range of the operator is the set of all vectors that could result from the mapping.

The MoM Procedure

Linear Operator: Examples

Example 1:
In a source-free homogeneous region, the time-harmonic electric field satisfies the equation *L*{**E**}=0, where *L* is the vector Helmholtz operator

$$L[\ ]=\left(\nabla^2+k^2\right)[\ ]$$

Example 2:
In an infinite homogeneous region with volume electric current distribution **J**(**r**), the time-harmonic electric field is given by the equation **E**(**r**)=*L*{**J**(**r**)}, where *L* is defined as

$$L\left[\mathbf{J}(\mathbf{r})\right]=-j\omega\mu\int_V\left[\mathbf{J}(\mathbf{r}')\right]g(\mathbf{r},\mathbf{r}')dv'$$

$$+\frac{1}{j\omega\varepsilon}\nabla\int_V\left\{\nabla'\cdot\left[\mathbf{J}(\mathbf{r}')\right]\right\}g(\mathbf{r},\mathbf{r}')dv'$$

The MoM Procedure

Basic Steps for Applying the MoM

- Formulation of the operator equation (typically using the SE)
- Selection of an appropriate “basis set”
- Selection of an appropriate “testing set”
- Enforcing equality of the projections
- Solving the matrix equation
- Computing relevant quantities

The MoM Procedure

Operator Equation Formulation

The operator equation will usually come directly from the physical situation or from some equivalent model.

The result is an operator equation of the form:

$$L[\mathbf{f}] = \mathbf{g}$$

Unknown Quantity (Current, Charge … etc.)

Known Quantity (Excitation or Boundary Conditions

The MoM Procedure

Selection of Appropriate Basis/Testing Set (Method of Least Squares)

$$L[\mathbf{f}]=\mathbf{g}$$

$$\mathbf{f}\cong\sum_{n=1}^{N}c_n\mathbf{f}_n=\tilde{\mathbf{f}}$$

Unknown Quantity Expansion (Basis)

$$\mathbf{g}=L[\mathbf{f}]\cong\sum_{n=1}^{N}c_nL[\mathbf{f}_n]=\tilde{\mathbf{g}}=\sum_{n=1}^{N}c_n\mathbf{w}_n$$

Known Quantity Expansion (Testing)

$$\mathbf{g}\cong\sum_{n=1}^{N}c_n\mathbf{w}_n=\tilde{\mathbf{g}}$$

$$\boxed{\mathbf{w}_n=L[\mathbf{f}_n]}$$

$$\langle\mathbf{g}-\tilde{\mathbf{g}},\mathbf{w}_k\rangle=0,\ k=0,1,\ldots,N$$

$$\langle\tilde{\mathbf{g}},\mathbf{w}_k\rangle=\langle\mathbf{g},\mathbf{w}_k\rangle,\ k=0,1,\ldots,N$$

The MoM Procedure

Selection of Appropriate Basis/Testing Set (Moment Method)

$$L[\mathbf{f}] = \mathbf{g}$$

$$\mathbf{f} \cong \sum_{n=1}^{N} c_n \mathbf{f}_n = \tilde{\mathbf{f}}$$

Unknown Quantity Expansion (Basis)

$$\mathbf{g} = L[\mathbf{f}] \cong \sum_{n=1}^{N} c_n L[\mathbf{f}_n] = \tilde{\mathbf{g}} = \sum_{n=1}^{N} d_n \mathbf{w}_n$$

$$\mathbf{g} \cong \sum_{n=1}^{N} d_n \mathbf{w}_n = \tilde{\mathbf{g}}$$

Known Quantity Expansion (Testing)

$$\boxed{\mathbf{w}_n \neq L[\mathbf{f}_n]}$$

$$\langle \mathbf{g} - \tilde{\mathbf{g}}, \mathbf{w}_k \rangle = 0,\ k = 0, 1, \ldots, N$$

$$\langle \tilde{\mathbf{g}}, \mathbf{w}_k \rangle = \langle \mathbf{g}, \mathbf{w}_k \rangle,\ k = 0, 1, \ldots, N$$

The MoM Procedure

Enforcing Equality of Projections

$$\left\langle \mathbf{w}_k, \sum_{n=1}^{N} c_n L[\mathbf{f}_n] \right\rangle = \langle \mathbf{w}_k, \mathbf{g} \rangle, \; k = 0, 1, \ldots, N$$

$$\begin{bmatrix} Z_{1,1} & Z_{1,2} & \cdots & Z_{1,N} \\ Z_{2,1} & Z_{2,2} & & Z_{2,N} \\ \vdots & & \ddots & \vdots \\ Z_{N,1} & Z_{N,2} & \cdots & Z_{N,N} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{bmatrix}$$

MoM Matrix Equation

$$Z_{kn} = \langle \mathbf{w}_k, L[\mathbf{f}_n] \rangle, \; V_k = \langle \mathbf{w}_k, \mathbf{g} \rangle$$

If the basis and testing functions are the same, this is called Galerkin’s projection technique.

Conclusion

- The MoM is a mathematical procedure for solving integral, differential or integro-differential equations by expanding the unknown function using a set of basis function with unknown coefficients.

- The result of the MoM is a matrix equation that should be solved to determine an approximate representation of the unknown function.