CUFE, M. Sc., 2015-2016

Computers & Numerical Analysis (STR 681)

Introduction

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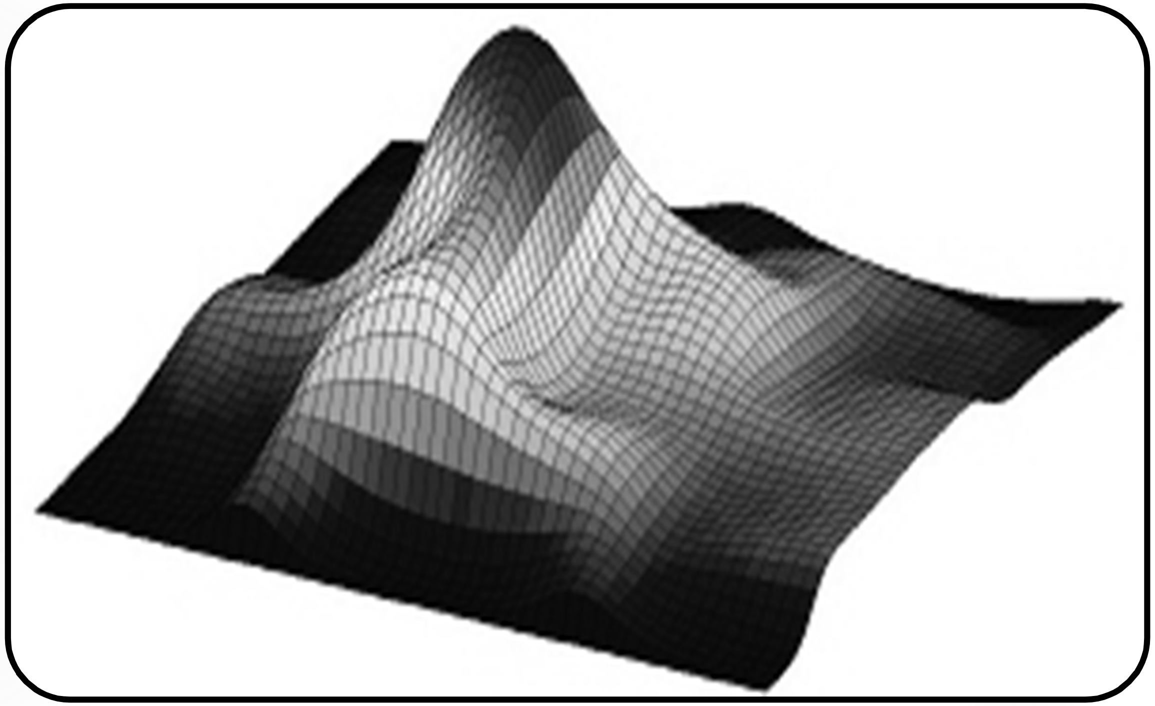
Why Do We Need Numerical Analysis Methods?



Introduction

- Numerical methods are techniques by which mathematical problems are formulated so that they can be solved with arithmetic operations.

- All numerical methods involves large numbers of arithmetic calculations.

Introduction

- There are three approaches for problem solving using non-computer methods.
    - ✓ Analytical (Exact Approach).
    - ✓ Graphical Solution.
    - ✓ Calculators and Slide Rules.

Introduction

1. Analytical (exact approach):

- Excellent insight into behavior.
- Derived for only a limited class of problems.
- Approximated for linear models, simple geometry, and low dimensionality.
- Limited practical value
- Real problems are nonlinear, complex, in shape and processes.

Introduction

2. Graphical solutions:

- Characterize the behavior of systems.
- Used to solve complex problems.
- Not very precise.
- Extremely tedious and awkward to implement.

Introduction

3. Calculators and Slide Rules:

- Implement numerical methods manually.
- Adequate for solving complex solutions.
- But slow and tedious.
- Consistency results are elusive: blunders.

Introduction

- Computers and numerical methods provide an alternative method for such calculations.

- Using computer power, problems can be approached without large simplifications or time-intense techniques.

Acknowledgement

THIS IS TO THANK DR. HESHAM SOBHY, DR. ASMAA HASSAN AND DR. AHMED AMIR BAYOUMY FOR GIVING THEIR VOLUNTARILY HELP IN PREPARING THIS PRESENTATION

Course Outline

Regulations

<u>إنش٦٨١: الحاسب والتحليل عددى</u>

مقدمة: البرمجة، حل المسائل، الخوارزميات، خريطة الانسياب، مقدمة عن التحليل الرقمى باستخدام الحاسب، تحليل الخطأ: أخطاء التمثيل والقطع والتقريب، المجموعات الخطية من المعادلات الجبرية: مشاكل الصفرية وسوء الضبط والدقة، طرق الإزالة: طرق حل المصفوفات الشريحية المتماثلة، مسألة القيم الذاتية: طريق الأس، طريقة جاكوبس، الطريقة المباشرة.

<u>STR681 Computer and Numerical Analysis</u>

Introduction: programming, problem solving, algorithm, flowcharting; Introduction to computer based numerical analysis; Error analysis: modeling, truncation, and round off errors; Linear sets of algebraic equations: singularity, ill-conditioning, and accuracy; Elimination techniques: banded and symmetric solvers, Eigen value problem: power method, Jacobi method, direct method.

Outline

- Systems of Linear Algebraic Equations
- Nonlinear Equations
- Polynomial Approximation & Interpolation
- Numerical Differentiation & Difference Formulas
- Numerical Integration
- Discretization & Finite Difference Methods

Outline

- Weighted Residual Approach
- Piecewise Functions
- Finite Element Methods
- Optimization
- Curve Fitting
- Ordinary Differential Equations
- Partial Differential Equations

Outline

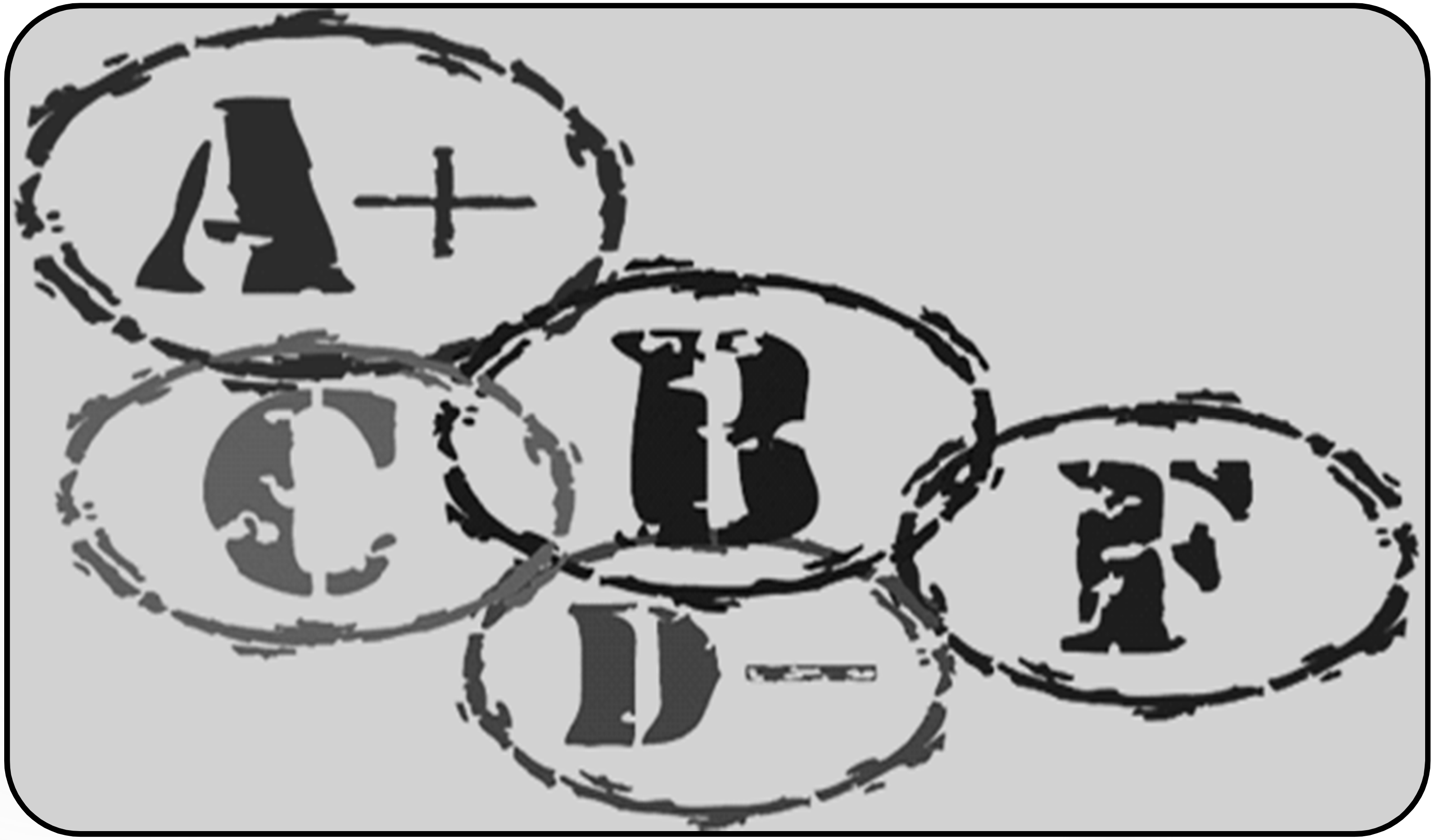
- Perturbation methods
- Fourier analysis
- Approximations & Round-off Errors

Outline

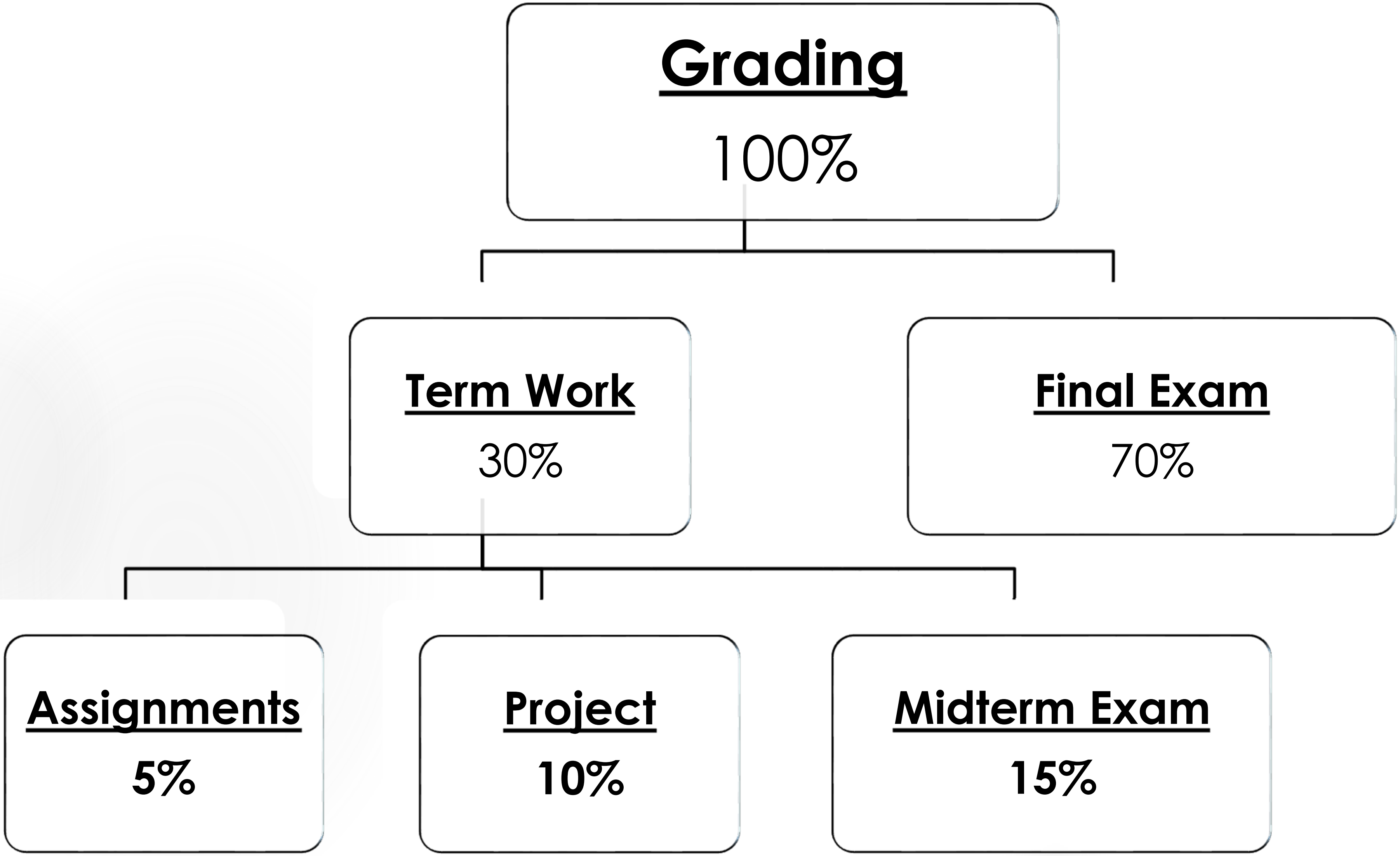
Programming
Language

MATLAB Program

Grading System



Grading System



Systems of Linear Algebraic Equations

$$
\begin{aligned}
a_{11}x_1 + \ a_{12}x_2 + \cdots + \ a_{1n}x_n &= 0 \\
a_{21}x_1 + \ a_{22}x_2 + \cdots + \ a_{2n}x_n &= 0 \\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0,
\end{aligned}
$$

Systems of Linear Algebraic Equations

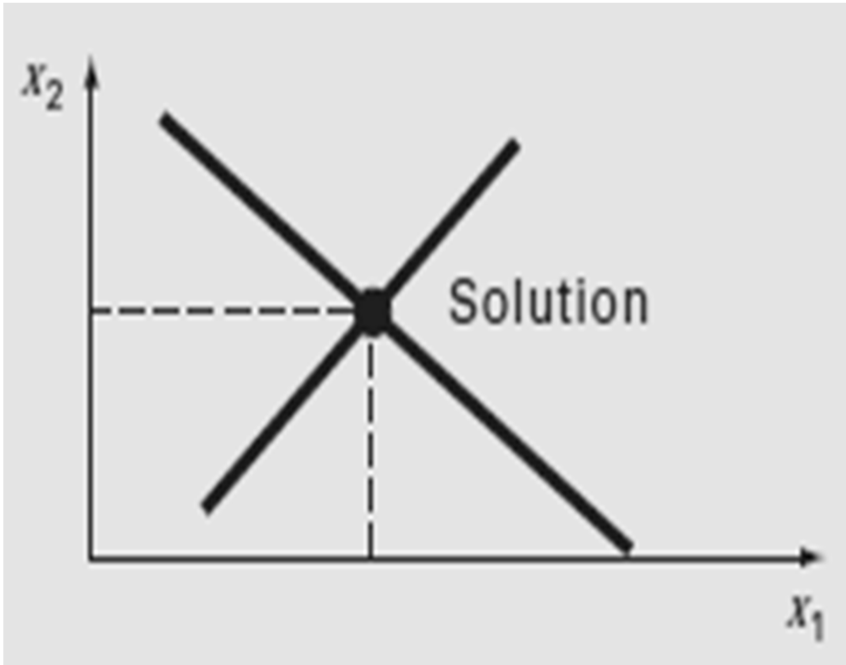
- Systems of linear algebraic equations:
  - These problems are concerned with the value of a set of variables that satisfies a set of linear equations.

Given the a's and the c's, solve

$$a_{11}x_1 + a_{12}x_2 = c_1$$

$$a_{21}x_1 + a_{22}x_2 = c_2$$

for the x's.

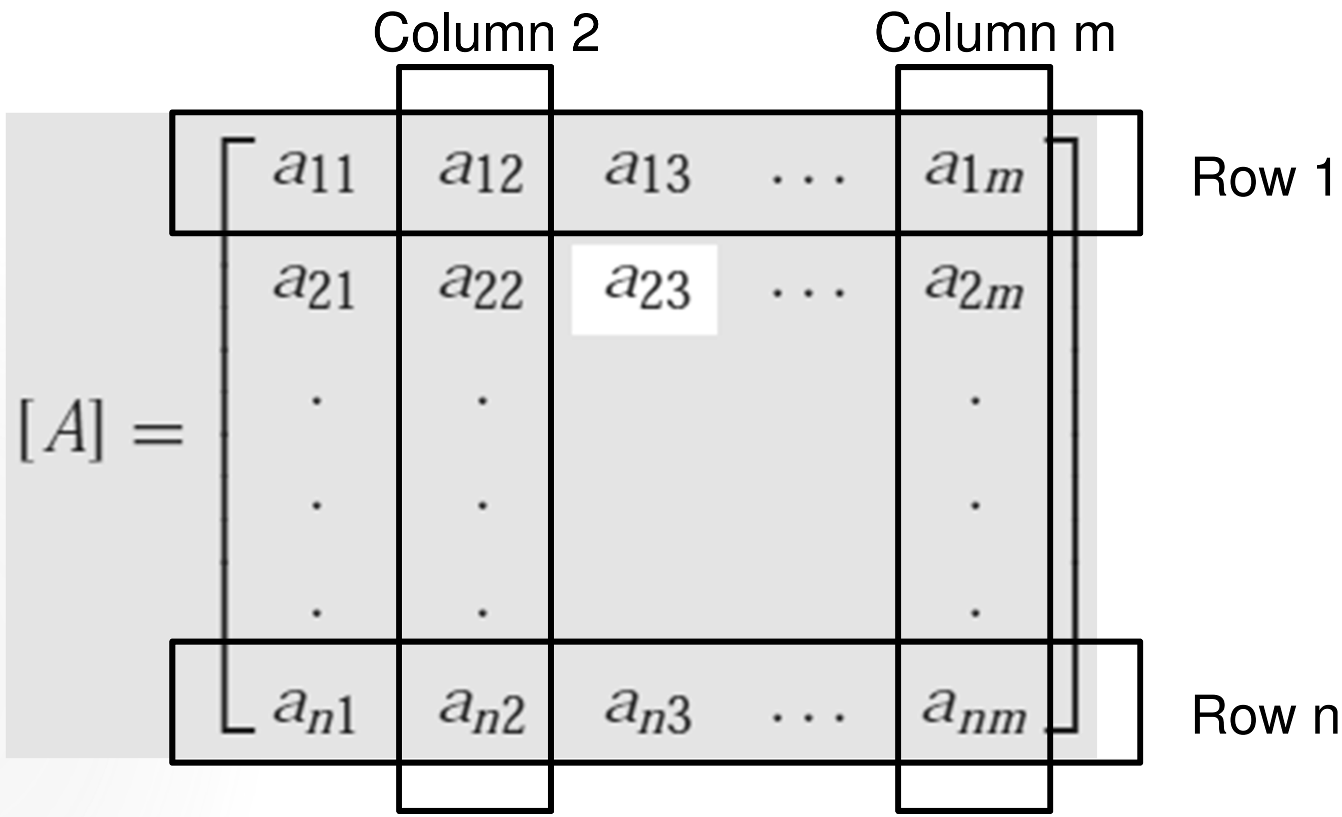


Systems of Linear Algebraic Equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\cdot \qquad\qquad\qquad & \cdot \\
\cdot \qquad\qquad\qquad & \cdot \\
\cdot \qquad\qquad\qquad & \cdot \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

$$[A]\{X\} = \{B\}$$

Matrix Notations



Matrix Notations

Row vector, n = 1

$$[B] = [b_1 \quad b_2 \quad \cdots \quad b_m]$$

Column vector, m = 1

$$[C] = \begin{bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{bmatrix}$$

Square matrix, n = m

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Matrix Notations

Symmetric matrix

$$[A] = \begin{bmatrix} 5 & 1 & 2 \\ 1 & 3 & 7 \\ 2 & 7 & 8 \end{bmatrix}$$

$$a_{ij} = a_{ji}$$

Diagonal matrix

$$[A] = \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & a_{33} & \\ & & & a_{44} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 11 \end{bmatrix}$$

Identity matrix

$$[I] = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix Notations

Upper triangular matrix

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ & a_{22} & a_{23} & a_{24} \\ & & a_{33} & a_{34} \\ & & & a_{44} \end{bmatrix} \quad \begin{bmatrix} 1 & 4 & 3 & 6 \\ 0 & 8 & 0 & 4 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 11 \end{bmatrix}$$

Lower triangular matrix

$$[A] = \begin{bmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ a_{31} & a_{32} & a_{33} & \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 8 & 0 & 0 \\ 7 & 2 & 1 & 0 \\ 2 & 7 & 6 & 11 \end{bmatrix}$$

Banded matrix

Ex., Band width = 3
(tridiagonal matrix)

$$[A] = \begin{bmatrix} a_{11} & a_{12} & & \\ a_{21} & a_{22} & a_{23} & \\ & a_{32} & a_{33} & a_{34} \\ & & a_{43} & a_{44} \end{bmatrix} \quad \begin{bmatrix} 1 & 4 & 0 & 0 \\ 5 & 8 & 3 & 0 \\ 0 & 2 & 1 & 6 \\ 0 & 0 & 6 & 11 \end{bmatrix}$$

Matrix Operating Rules

$$[A] = [B] \text{ if } a_{ij} = b_{ij} \text{ for all } i \text{ and } j.$$

$$\begin{bmatrix} 1 & 4 & 5 \\ 7 & 12 & 2 \\ 8 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 5 \\ 7 & 12 & 2 \\ 8 & 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 \\ 7 & 12 & 2 \\ 8 & 3 & \boxed{4} \end{bmatrix} \neq \begin{bmatrix} 1 & 4 & 5 \\ 7 & 12 & 2 \\ 8 & 3 & \boxed{10} \end{bmatrix}$$

Matrix Operating Rules

$[A] + [B] =$

$c_{ij} = a_{ij} + b_{ij}$

$$\begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 8 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 12 \\ 2 & 15 \end{bmatrix}$$

$d_{ij} = e_{ij} - f_{ij}$

$$\begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 8 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 2 & -3 \end{bmatrix}$$

Matrix Operating Rules

$$\begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 8 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 12 \\ 2 & 15 \end{bmatrix}$$

$$[A] + [B] = [B] + [A]$$

$$\begin{bmatrix} 2 & 8 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 12 \\ 2 & 15 \end{bmatrix}$$

$$([A] + [B]) + [C] = [A] + ([B] + [C])$$

Matrix Operating Rules

$$[D] = g[A] = \begin{bmatrix} ga_{11} & ga_{12} & \cdots & ga_{1m} \\ ga_{21} & ga_{22} & \cdots & ga_{2m} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ ga_{n1} & ga_{n2} & \cdots & ga_{nm} \end{bmatrix}$$

$$2x \begin{bmatrix} 1 & 4 & 5 \\ 7 & 12 & 2 \\ 8 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 8 & 10 \\ 14 & 24 & 4 \\ 16 & 6 & 8 \end{bmatrix}$$

Matrix Operating Rules

$$[C] = [A][B]$$

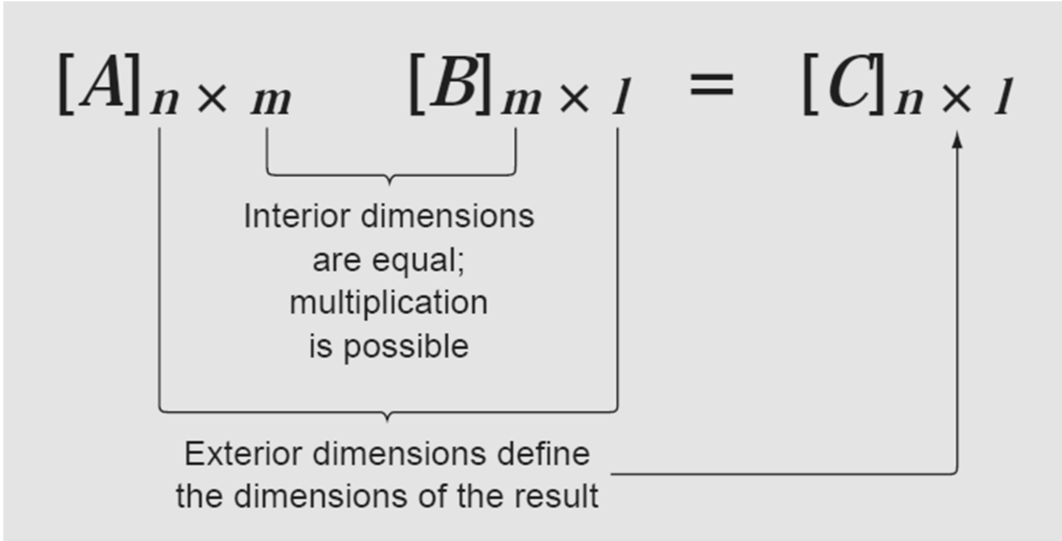
$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$[A]$ is an n by $\boxed{m}$ matrix, $[B]$ could be an $\boxed{m}$ by l matrix

column row

Matrix Operating Rules

$$[A][B] = [C]$$



$$\begin{bmatrix} 3 & 1 \\ 8 & 6 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 5 & 9 \\ 7 & 2 \end{bmatrix} = [?]$$

3 by 2

2 by 2

3 by 2

Matrix Operating Rules

$$\begin{bmatrix} 3 & 1 \\ 8 & 6 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 5 & 9 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 3x5 + 7x1 & ? \\ ? & ? \\ ? & ? \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 8 & 6 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 5 & 9 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 22 & 3x9 + 1x2 \\ ? & ? \\ ? & ? \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 8 & 6 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 5 & 9 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 22 & 29 \\ 8x5 + 6x7 & ? \\ ? & ? \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 8 & 6 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 5 & 9 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 22 & 29 \\ 82 & 84 \\ 28 & 8 \end{bmatrix}$$

Matrix Operating Rules

$$([A][B])[C] = [A]([B][C])$$

$$[A]([B] + [C]) = [A][B] + [A][C]$$

$$([A] + [B])[C] = [A][C] + [B][C]$$

$$\boxed{[A][B] \neq [B][A]}$$

Matrix Operating Rules

If A is a square matrix: $[A][A]^{-1} = [A]^{-1}[A] = [I]$

A^{-1} is the inverse of A

$$[A]^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$\text{Ex., } A = \begin{bmatrix} 5 & 9 \\ 7 & 2 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{5x2 - 9x7} \begin{bmatrix} 2 & -9 \\ -7 & 5 \end{bmatrix}$$

Matrix Operating Rules

If A is a square matrix: $[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$

A^T is the transpose of A $[A]^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} \\ a_{13} & a_{23} & a_{33} & a_{43} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix}$

$$\text{Ex., } A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 2 & 5 & 8 \\ 3 & 6 & 9 \\ 4 & 7 & 10 \end{bmatrix}$$

Matrix Operating Rules

$$\text{tr}\,[A] = \sum_{i=1}^{n} a_{ii}$$

tr [A] is the trace of matrix [A]

$$\text{Ex., A} = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix} \Rightarrow \text{tra}[A] = 2 + 6 + 10 = 18$$

Matrix Operating Rules

For a 2 x 2 determinant $D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

Ex., $D = \begin{vmatrix} 3 & 2 \\ -1 & 2 \end{vmatrix} = 3(2) - 2(-1) = 8$

Matrix Operating Rules

For a 3 x 3 determinant $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

minor

$$D = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Ex., $D = \begin{vmatrix} 0.3 & 0.52 & 1 \\ 0.5 & 1 & 1.9 \\ 0.1 & 0.3 & 0.5 \end{vmatrix}$

D = 0.3 x $\begin{vmatrix} 1 & 1.9 \\ 0.3 & 0.5 \end{vmatrix}$ - 0.52 x $\begin{vmatrix} 0.5 & 1.9 \\ 0.1 & 0.5 \end{vmatrix}$ + 1 x $\begin{vmatrix} 0.5 & 1 \\ 0.1 & 0.3 \end{vmatrix}$

D = 0.3 x (1x0.5 – 0.3 x 1.9) – 0.52 x 0.06 + 1 x 0.05 = -0.0022

Matrix Operating Rules

"Augmentation"

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow [A] = \left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right]$$

Such an expression has utility when we must perform a set of identical operations on two matrices. Thus, we can perform the operations on the single augmented matrix rather than on the two individual matrices.

Matrix Operating Rules

$$[A]\{X\} = \{B\}$$

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\{X\}^T = \lfloor x_1 \quad x_2 \quad \cdots \quad x_n \rfloor$$

$$\{B\}^T = \lfloor b_1 \quad b_2 \quad \cdots \quad b_n \rfloor$$

$$\{X\} = \frac{\{B\}}{[A]}$$

No division in Matrices

Matrix Operating Rules

$$[A]\{X\} = \{B\}$$

$$[A]^{-1}[A]\{X\} = [A]^{-1}\{B\}$$

$$\boxed{[A]^{-1}[A] = [I]}$$

$$\{X\} = [A]^{-1}\{B\}$$

Solving Linear Algebraic Equations

- Solving small number of equations:
  - ✓ The Graphical Method.
  - ✓ Cramer's Rule.
  - ✓ The Elimination of Unknowns.
- Gauss Elimination
- Gauss-Jordan
- LU Decomposition
- Gauss Seidel

Solving Linear Algebraic Equations

- **Direct Methods:**
  - ✓ **Cramer's Rule.**
  - ✓ **Gauss Elimination.**
  - ✓ **Gauss Jordan.**
  - ✓ **Banded Matrix.**
  - ✓ **Skyline**
- **Iterative Methods**
  - ✓ **Jacobi Iteration.**
  - ✓ **Gauss-Seidel**

The Graphical Method

Suitable for small number of equations (≤ 3)

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

⇒

$$x_2 = \boxed{-\left(\frac{a_{11}}{a_{12}}\right)} x_1 + \boxed{\frac{b_1}{a_{12}}}$$
$$x_2 = \boxed{-\left(\frac{a_{21}}{a_{22}}\right)} x_1 + \boxed{\frac{b_2}{a_{22}}}$$

slope intercept

The Graphical Method

Ex., Solve

$$3x_1 + 2x_2 = 18$$
$$-x_1 + 2x_2 = 2$$

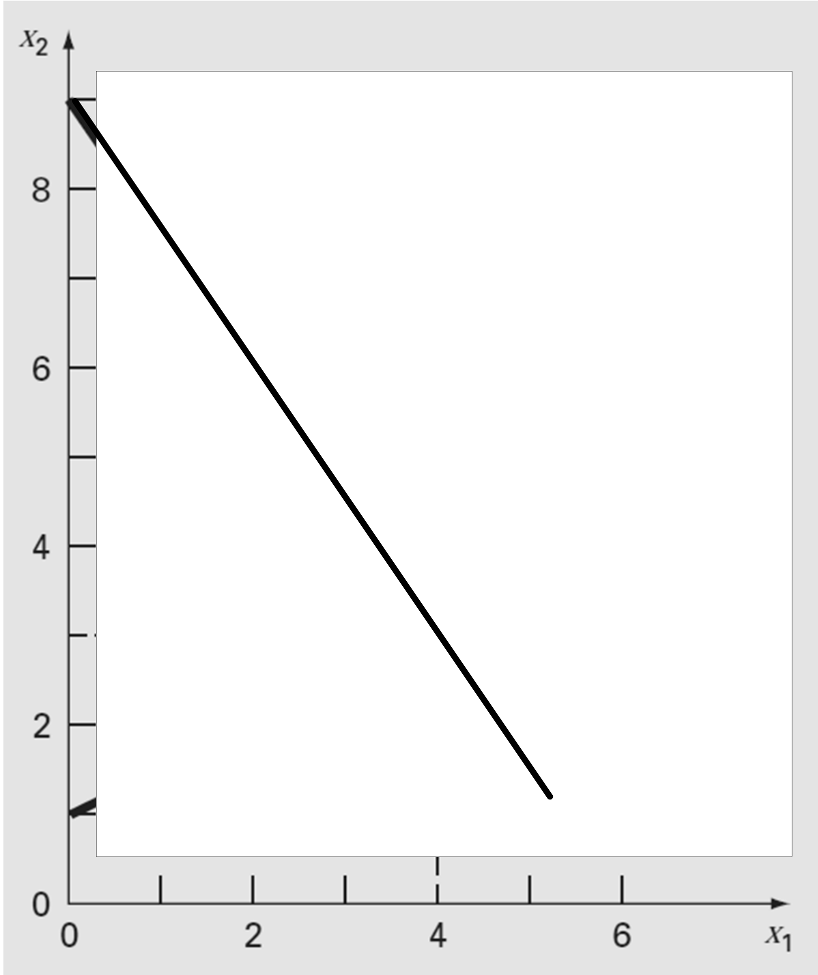
$$3x_1 + 2x_2 = 18$$

⇓

$$x_2 = \boxed{-\frac{3}{2}x_1} + \boxed{9}$$

intercept

slope



The Graphical Method

Ex., Solve

$$3x_1 + 2x_2 = 18$$
$$-x_1 + 2x_2 = 2$$

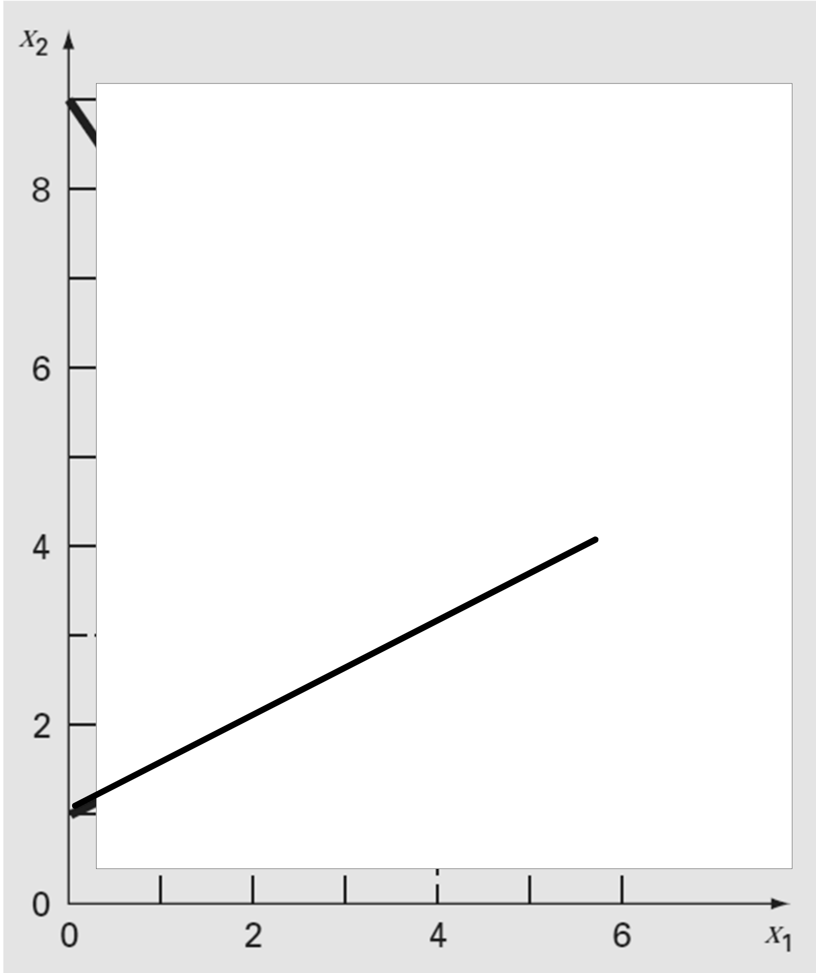
$$-x_1 + 2x_2 = 2$$

⇓

$$x_2 = \boxed{\frac{1}{2}} x_1 + \boxed{1}$$

intercept

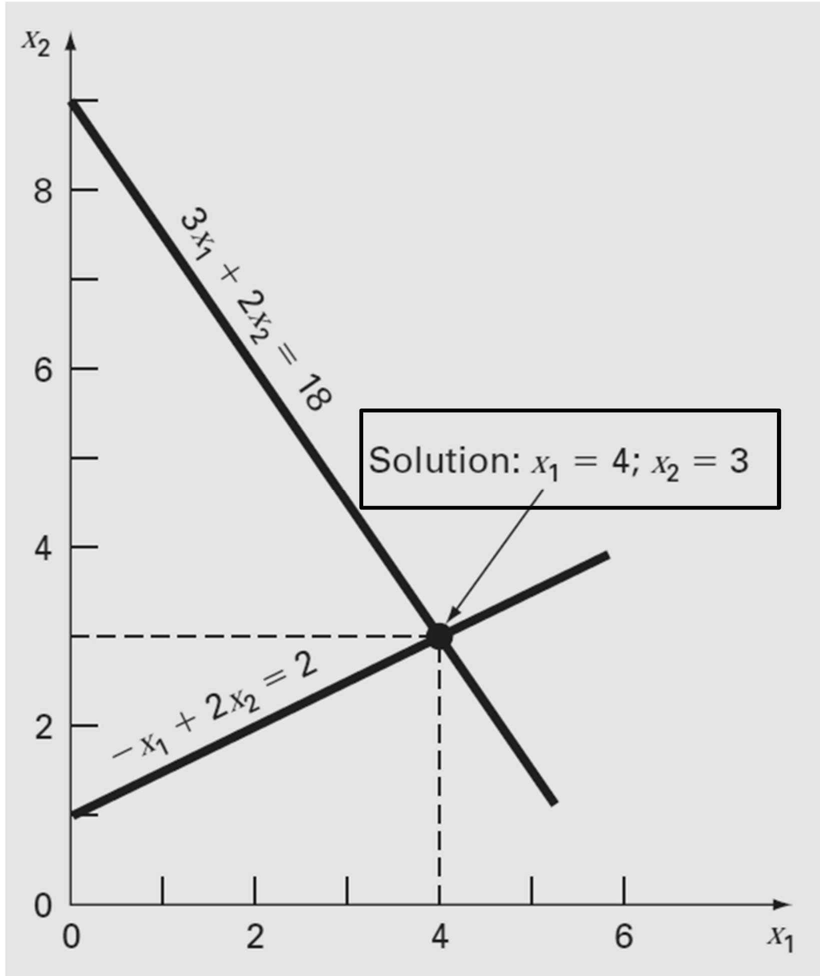
slope



The Graphical Method

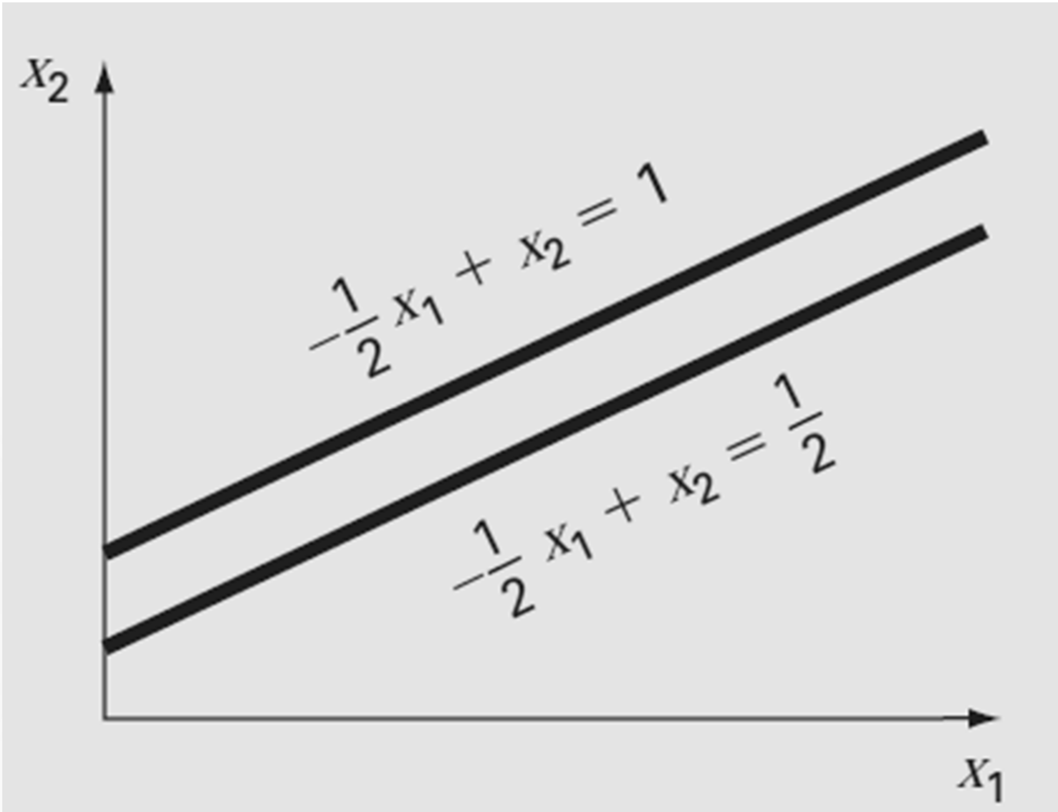
Ex., Solve

$$3x_1 + 2x_2 = 18$$
$$-x_1 + 2x_2 = 2$$



The Graphical Method

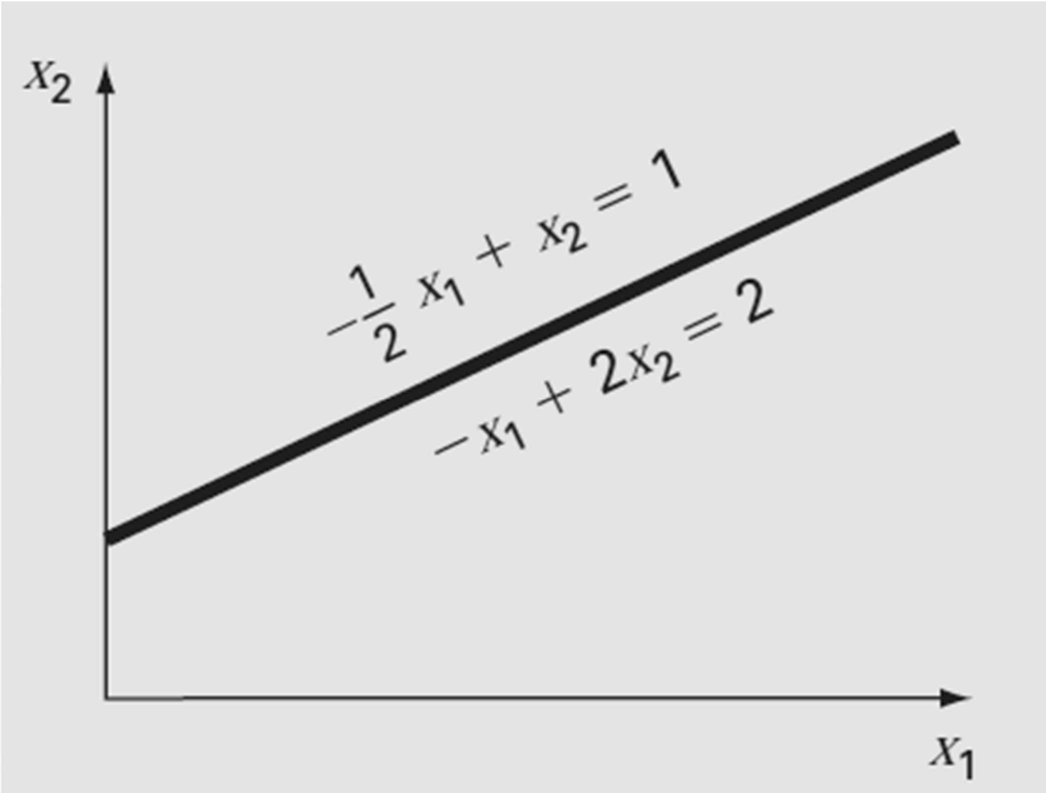
Is there cases where there will be no solution?



Parallel Lines : No solution
(Singular system)

The Graphical Method

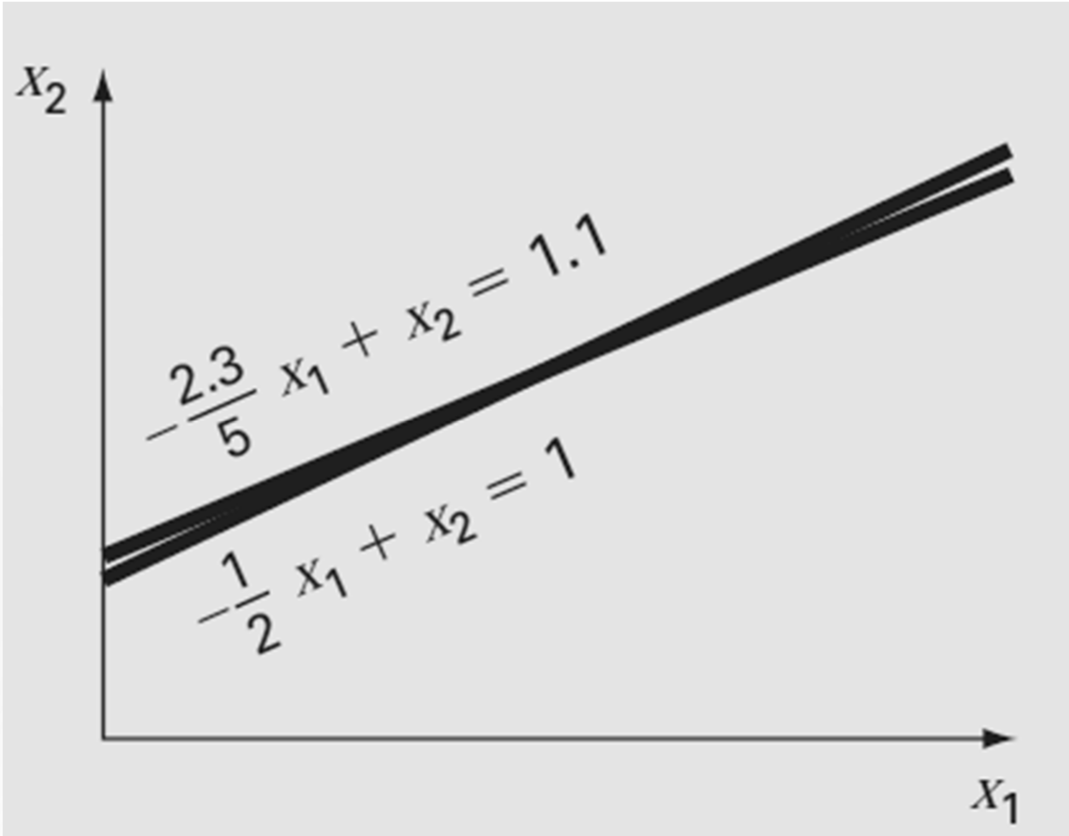
Is there cases where there will be no solution?



Coincident Lines : Infinite solutions
(Singular system)

The Graphical Method

Is there cases where there will be no solution?



Ill conditioned system

Cramer's Rule

This rule states that each unknown in a system of linear algebraic equations may be expressed as a fraction of two determinants with denominator D and with the numerator obtained from D by replacing the column of coefficients of the unknown in question by the constants $b_1, b_2, \ldots, b_n$. For example, x_1 would be computed as:

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{D}$$

Cramer's Rule

Ex.,
$$0.3x_1 + 0.52x_2 + x_3 = -0.01$$
$$0.5x_1 + x_2 + 1.9x_3 = 0.67$$
$$0.1x_1 + 0.3x_2 + 0.5x_3 = -0.44$$

$$D = \begin{vmatrix} 0.3 & 0.52 & 1 \\ 0.5 & 1 & 1.9 \\ 0.1 & 0.3 & 0.5 \end{vmatrix} = 0.3(-0.07) - 0.52(0.06) + 1(0.05) = -0.0022$$

Cramer's Rule

Ex., $0.3x_1 + 0.52x_2 + x_3 = -0.01$

$0.5x_1 + x_2 + 1.9x_3 = 0.67$

$0.1x_1 + 0.3x_2 + 0.5x_3 = -0.44$

$$x_1 = \frac{\begin{vmatrix} -0.01 & 0.52 & 1 \\ 0.67 & 1 & 1.9 \\ -0.44 & 0.3 & 0.5 \end{vmatrix}}{-0.0022} = \frac{0.03278}{-0.0022} = -14.9$$

Cramer's Rule

Ex., $0.3x_1 + 0.52x_2 + x_3 = -0.01$

$0.5x_1 + x_2 + 1.9x_3 = 0.67$

$0.1x_1 + 0.3x_2 + 0.5x_3 = -0.44$

$$x_2 = \frac{\begin{vmatrix} 0.3 & -0.01 & 1 \\ 0.5 & 0.67 & 1.9 \\ 0.1 & -0.44 & 0.5 \end{vmatrix}}{-0.0022} = \frac{0.0649}{-0.0022} = -29.5$$

Cramer's Rule

Ex., $0.3x_1 + 0.52x_2 + x_3 = -0.01$

$0.5x_1 + x_2 + 1.9x_3 = 0.67$

$0.1x_1 + 0.3x_2 + 0.5x_3 = -0.44$

$$x_3 = \frac{\begin{vmatrix} 0.3 & 0.52 & -0.01 \\ 0.5 & 1 & 0.67 \\ 0.1 & 0.3 & -0.44 \end{vmatrix}}{-0.0022} = \frac{-0.04356}{-0.0022} = 19.8$$

The Elimination of Unknowns

- The basic strategy is to multiply the equations by constants so that one of the unknowns will be eliminated when the two equations are combined.
- The result is a single equation that can be solved for the remaining unknown.
- This value can then be substituted into either of the original equations to compute the other variable.

The Elimination of Unknowns

$$a_{11}x_1 + a_{12}x_2 = b_1$$ multiply by a_{21}

$$a_{21}x_1 + a_{22}x_2 = b_2$$ multiply by a_{11}

$$a_{11}a_{21}x_1 + a_{12}a_{21}x_2 = b_1 a_{21}$$

$$a_{21}a_{11}x_1 + a_{22}a_{11}x_2 = b_2 a_{11}$$

$$a_{22}a_{11}x_2 - a_{12}a_{21}x_2 = b_2 a_{11} - b_1 a_{21}$$

$$x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$$

$$x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

Cramer's rule

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

The Elimination of Unknowns

Ex., $3x_1 + 2x_2 = 18$

$-x_1 + 2x_2 = 2$

$$x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}} = \frac{2(18) - 2(2)}{3(2) - 2(-1)} = 4$$

$$x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}} = \frac{3(2) - (-1)18}{3(2) - 2(-1)} = 3$$

Gauss Elimination

The procedure consisted of two steps:

1. **The equations were manipulated to eliminate one of the unknowns from the equations. The result of this *elimination* step was that we had one equation with one unknown.**
2. **Consequently, this equation could be solved directly and the result *back-substituted* into one of the original equations to solve for the remaining unknown.**

Gauss Elimination

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ & a'_{22} & a'_{23} & b'_2 \\ & & a''_{33} & b''_3 \end{array}\right]$$

Forward elimination

$$\Downarrow$$

$$x_3 = b''_3 / a''_{33}$$
$$x_2 = (b'_2 - a'_{23} x_3) / a'_{22}$$
$$x_1 = (b_1 - a_{12} x_2 - a_{13} x_3) / a_{11}$$

Back substitution

Gauss Elimination

Ex.,

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$
$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$
$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 3 & -0.1 & -0.2 & 7.85 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.2 & 10 & 71.4 \end{array}\right]$$

Gauss Elimination

Pivot

$$\left[\begin{array}{ccc|c} 3 & -0.1 & -0.2 & 7.85 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.2 & 10 & 71.4 \end{array}\right]$$

x 0.1/3

$$\left[\begin{array}{ccc|c} 3 & -0.1 & -0.2 & 7.85 \\ 0 & 7.00333 & -0.29333 & -19.5617 \\ 0.3 & -0.2 & 10 & 71.4 \end{array}\right]$$

x 0.3/3

$$\left[\begin{array}{ccc|c} 3 & -0.1 & -0.2 & 7.85 \\ 0 & 7.00333 & -0.29333 & -19.5617 \\ 0 & -0.19 & 10.02 & 70.615 \end{array}\right]$$

Pivot

x -0.19/7.003333

Gauss Elimination

$$\left[\begin{array}{ccc:c} 3 & -0.1 & -0.2 & 7.85 \\ 0 & 7.00333 & -0.29333 & -19.5617 \\ 0 & 0 & 10.0120 & 70.0843 \end{array}\right]$$

$$x_3 = \frac{70.0843}{10.0120} = 7.0000$$

Gauss Elimination

$$\left[\begin{array}{ccc|c} 3 & -0.1 & -0.2 & 7.85 \\ 0 & 7.00333 & -0.29333 & -19.5617 \\ 0 & 0 & 10.0120 & 70.0843 \end{array}\right]$$

$$7.00333x_2 - 0.293333(7.0000) = -19.5617$$

$$x_2 = \frac{-19.5617 + 0.293333(7.0000)}{7.00333} = -2.50000$$

Gauss Elimination

$$\left[\begin{array}{ccc:c} 3 & -0.1 & -0.2 & 7.85 \\ 0 & 7.00333 & -0.29333 & -19.5617 \\ 0 & 0 & 10.0120 & 70.0843 \end{array}\right]$$

$$3x_1 - 0.1(-2.50000) - 0.2(7.0000) = 7.85$$

$$x_1 = \frac{7.85 + 0.1(-2.50000) + 0.2(7.0000)}{3} = 3.00000$$

Drawbacks of Gauss Elimination

Dividing by zero (or close to zero)

- **Solution: "Partial pivoting" switch rows and use the largest value as a pivot**

Pivot=0

$$
\begin{aligned}
2x_2 + 3x_3 &= 8 \\
4x_1 + 6x_2 + 7x_3 &= -3 \\
2x_1 + x_2 + 6x_3 &= 5
\end{aligned}
$$

Drawbacks of Gauss Elimination

Round-off errors

Important for more than 100 equations

You should always substitute your answers back into the original equations to check whether a substantial error has occurred.

Solution: scaling & use more significant fraction figures

Drawbacks of Gauss Elimination

Ex., $2x_1 + 100,000x_2 = 100,000$
$x_1 + x_2 = 2$

Correct answer

$$x_1 = 1.00002, x_2 = 0.99998$$

$$\left[\begin{array}{cc|c} 2 & 100000 & 100000 \\ 1 & 1 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 2 & 100000 & 100000 \\ 0 & -49999 & -49998 \end{array}\right]$$

$$x_2 = \frac{-49998}{-49999} = 0.99998 \cong 1$$

$$2x_1 + 100,000x1 = 100,000 \Rightarrow x_1 = 0 \times$$

Drawbacks of Gauss Elimination

- Use more significant fraction figures

$$\left[\begin{array}{cc:c} 2 & 100000 & 100000 \\ 1 & 1 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{cc:c} 2 & 100000 & 100000 \\ 0 & -49999 & -49998 \end{array}\right]$$

$$x_2 = \frac{-49998}{-49999} = 0.99998$$

$$2x_1 + 100{,}000x0.99998 = 100{,}000 \Rightarrow x_1 = 1$$

Drawbacks of Gauss Elimination

Use scaling that maximum coefficient in each row is 1

$$\left[\begin{array}{cc|c} 0.00002 & 1 & 1 \\ 1 & 1 & 2 \end{array}\right] \xrightarrow{\text{Partial pivoting}} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0.00002 & 1 & 1 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0.99998 & 0.99996 \end{array}\right]$$

$$\Leftarrow$$

$$x_2 = \frac{0.99996}{0.99998} = 0.99998$$

$$x_1 = 1.00002$$

Drawbacks of Gauss Elimination

3. Singular systems (D = 0)

4. Ill conditioned systems

- Ill-conditioning is that a wide range of answers can approximately satisfy the equations.

- Because round-off errors can induce small changes in the coefficients, these artificial changes can lead to large solution errors for ill-conditioned systems.

- $D \approx 0$ (scaling numbers in the matrix to be less than one)

- Solution: use more significant figures

Drawbacks of Gauss Elimination

Ex., $x_1 + 2x_2 = 10$

$1.1x_1 + 2x_2 = 10.4$

$$\left[\begin{array}{cc|c} 1 & 2 & 10 \\ 1.1 & 2 & 10.4 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 2 & 10 \\ 0 & -0.2 & -0.6 \end{array}\right]$$

$$x_2 = \frac{-0.6}{-0.2} = 3$$

$$x_1 + 2x3 = 10 \Rightarrow x_1 = 4$$

Drawbacks of Gauss Elimination

Replace 1.1 by 1.05

$$\left[\begin{array}{cc:c} 1 & 2 & 10 \\ 1.05 & 2 & 10.4 \end{array}\right] \Rightarrow \left[\begin{array}{cc:c} 1 & 2 & 10 \\ 0 & -0.1 & -0.1 \end{array}\right]$$

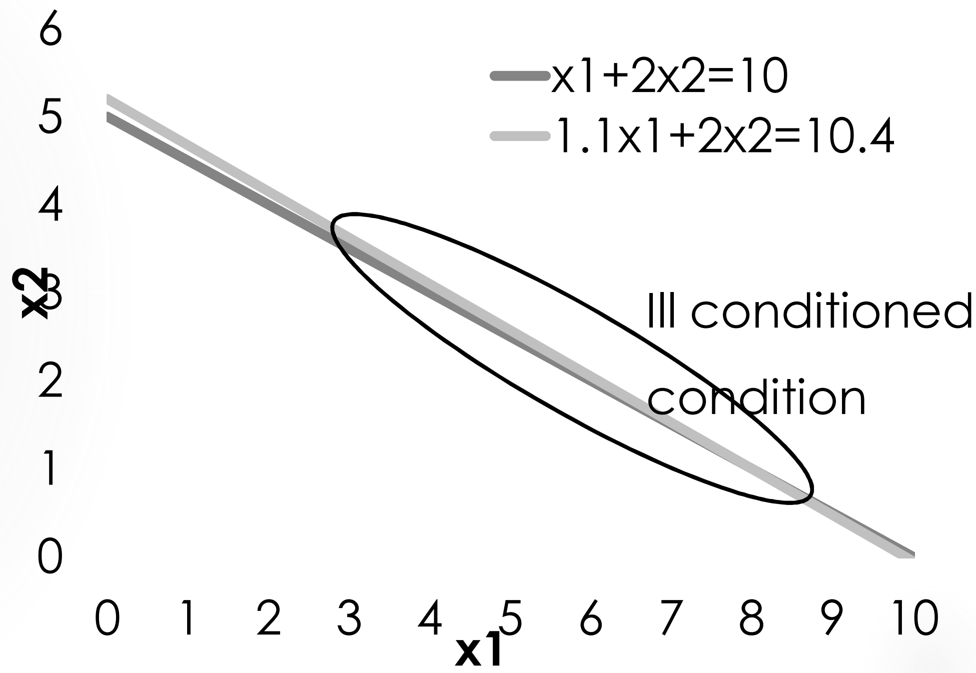
$$x_2 = \frac{-0.1}{-0.1} = 1$$

$$x_1 + 2x1 = 10 \quad \Rightarrow \quad x_1 = 8$$

Drawbacks of Gauss Elimination

Replace 1.1 by 1.05

$$x_1 = 4, x_2 = 3 \qquad x_1 = 8, x_2 = 1$$



Gauss Elimination

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ & a'_{22} & a'_{23} & b'_2 \\ & & a''_{33} & b''_3 \end{array}\right]$$

Forward elimination

$$\Downarrow$$

$$x_3 = b''_3 / a''_{33}$$

$$x_2 = (b'_2 - a'_{23} x_3) / a'_{22}$$

$$x_1 = (b_1 - a_{12} x_2 - a_{13} x_3) / a_{11}$$

Back substitution